The Formula of the Frobenius Number for a Numerical Semigroup with Embedding Dimension Three respect to a Partial Order Relation

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Abstract

The Frobenius number is the largest positive integer that cannot be expressed as a non-negative linear combination of a given set of positive integers. It is considered to be one of the well-known problems in number theory, especially where the cardinality of the set is greater than three. Recently, algorithms and formulas have been proposed to calculate the Frobenius number in three variables, and various techniques have been used to handle the problem. Currently, for more than three variables, the problem associated with finding the Frobenius number is still considered to be an open problem. In this work, we used the concept of a numerical semigroup to develop an alternative approach to finding the Frobenius and genus number in three variables in particular cases. In the arbitrary variables, the formula presented in three variables can yield an upper bound of the Frobenius number and genus number.

Keywords: Embedding dimension, Frobenius number, Genus number, Numerical semigroup, Partial order relation

Introduction

During the 19^{th} century, Ferdinand Georg Frobenius (1849-1917), who invented the Frobenius method used for solving a second-order ordinary differential equation, proposed the problem of finding the largest positive integer where it cannot be a solution of a liner equation with given non-negative coefficients. This largest positive number is called the *Frobenius number*, and the problem related to this is called the Frobenius problem (Ramirez Alfonsin (2005)). With the same given constraint, the number of positive integers that cannot be written in the form of the linear combination is called the *genus number*. For example, given a linear equation $3x + 4y$ where x, y is constrained to be non-negative integers, all positive integers that cannot be expressed by this equation are 1,2,5. Hence, 5 is the Frobenius number, and 3 is a genus number. In two variables case, the problem associated with finding formulae for the genus number and the Frobenius number seems not too complicated. In fact, the exact formulae in two variables case were proposed in 1884 by James Joseph Sylvester (1884). However, the Frobenius problem becomes much more complicated once the number of variables is greater than two. Unfortunately, Curtis (1990) proved that for three or more variables, there is no such formula for finding the Frobenius in polynomial terms that can be computed.

Since the 1950s, the concept of numerical semigroups gained interest due to their application in algebraic geometry (Barucci, Dobbs, & Fontana, 1997). According to Ramirez Alfonsin (2005), the Frobenius problem can be viewed as the coin problem, which is about finding the largest amount of money that is unable to be obtained after the specific coins were given. This can be linked to the concept of numerical semigroups where every possible number that is obtained by the given coins belongs to a numerical semigroup. To give a specific detail, for notations and definitions which always used throughout this research, the notation N denotes the set of all positive integers and $\mathbb{N}^0$ denotes the set of positive integers including zero. According to Nari, Numata, and Watanabe (2012), for a set S with $\emptyset \neq S \subseteq \mathbb{N}^0$ under a usual addition is a *numerical semigroup* if it satisfies the following conditions:

$0 \in S$ as well as S is closed, and $\mathbb{N}^0 \backslash S$ is finite. In addition, for a numerical semigroup S, let A be a finite subset of $\mathbb{N}$. The notations $\langle A \rangle$ is defined as the set of all possible linear combinations of all elements in A. In other words,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n : n \in \mathbb{N},\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}^0 \text{ and } a_1, \ldots, a_n \in A\}.$$

If $\langle A \rangle = S$, then we say that S is generated by A, and A is called a *system of generators* of S. To show that $\langle A \rangle$ is a numerical semigroup, the simple way that Rosales and Garcia-Sanchez (2009) suggested is that it is a numerical semigroup if and only if there exists $a, b \in A$ and $a \neq b$ such that $\gcd(a, b) = 1$. In addition, if for each $a \in A$ is unable to be represented as a linear combination of elements in A over $\mathbb{N}^0$ excepts a itself, then A is the *minimal system of generators* of S. In this case, $|A|$ can be called the *embedding dimension* of S. Also, Rosales and Garcia-Sanchez (2009) showed that every numerical semigroup has the unique system of generators. In particular, if $A = \{a_1, \ldots, a_n\}$, $\langle A \rangle$ can be conveniently denoted by $\langle a_1, \ldots, a_n \rangle$.

Using the notions suggested above, the rigorous can be formulated as followings: the genus number of a numerical semigroup S is the cardinality of $\mathbb{N}^0 \backslash S$ which is denoted by $g(S)$, and the Frobenius number of a numerical semigroup S is denoted as the largest element of $\mathbb{N}^0 \backslash S$ using a notation $F(S)$. With the connection to a concept of numerical semigroup, the Frobenius problem mentioned before can be formulated as finding the Frobenius number of $\langle a_1, \ldots, a_n \rangle$ where $\gcd(a_1, \ldots, a_n) = 1$. The simple exact formula for the two variables case was proposed by Sylvester (1884), that is,

$$F(\langle a_1, a_2 \rangle) = a_1 a_2 - a_1 - a_2 \text{ and } g(\langle a_1, a_2 \rangle) = \frac{1}{2}(a_1 a_2 - a_1 - a_2 + 1). \qquad (*)$$

Almost a century after Sylvester discovered the formula, it took about a century when ways of finding the exact solution of Frobenius number in three variables are discovered. Currently, many forms of formula for the three formulas have been proposed. Specifically, algorithms and methods of finding the Frobenius number in three variables were presented, such as Selmer and Beyer (1978), Rodseth (1978), and Davison (1994). Even though Curtis showed that no such formulas in the form of polynomial can be used for finding the Frobenius number, these methods provide the solution with the algorithmic approach. Recently, Tripathi (2017) has proposed formulas which can be used to calculate the Frobenius number in three variables. Although the formulas are known for some subcases, his work is based on filling the gap of the unknown cases to present the complete formula. For the three variables case, some formulas in some subcases are presented, such as Gu (2020), and Gu et al. (2022). For arbitrary cases, it is noticeable that the formula presented by Johnson (1960) showed the relationship between the Frobenius number of the numerical semigroup generated by$\{a_1, a_2, \ldots, a_{n-1}, a_n\}$ and $\{\frac{a_1}{d}, \frac{a_2}{d}, \ldots, \frac{a_{n-1}}{d}, a_n\}$ where $\gcd(a_1, \ldots, a_{n-1}) = d$. Instead of finding the Frobenius number directly, the formula can be helpful to transform the problem into the generator with lesser coefficients, which is $\{\frac{a_1}{d}, \frac{a_2}{d}, \ldots, \frac{a_{n-1}}{d}, a_n\}$. However, the formula cannot be used to directly calculate the Frobenius number, and according to Gu et al. (2022), the general exact formula for the more variables case is still an open problem.

In this work, we focus on the finding of alternative exact formulas for three variables in some subcases using the concept of *pseudo-Frobenius number*, which is numbers behaves like the Frobenius number, but it is possible to have more than one element. It is suggested in Froberg et al. (1987) that for a numerical semigroup S, a partial order relation $\preccurlyeq_S$ can be formed on $\mathbb{Z}$ with the concept of the pseudo-Frobenius number. However, this connection is fruitful because of the property for the numerical semigroup S with embedding dimension two where the set of pseudo-Frobenius number has the Frobenius number as its only element. Furthermore, we will propose the alternative exact formula of $g(S)$ and $F(S)$ in a particular case and the upper bound of $g(S)$ and $F(S)$ in the general cases where $\{a, b, c\}$ with $a < b < c$ is the minimal system of generator of S.

Preliminaries

In this section, we will discuss and establish the rigorous definitions, notations, and more details of all key theorems, which will be used throughout this research. The following definitions and theorems are the required key to obtain results in this research.

Theorem 1 (Rosales & Garcia-Sanchez, 2009). $S = \langle a_1, \ldots, a_n \rangle$ is a numerical semigroup if and only if $gcd(a_1, \ldots, a_n) = 1$ where $a_1, \ldots, a_n \in \mathbb{N}$.

If a generator $\{a_1, \ldots, a_n\}$ of a given set S is provided, instead of directly showing that S is a numerical semigroup by its definition, the Theorem 1 can be useful to check whether the given set is a numerical semigroup by easily checking its greatest common divisor of the generator. The equivalent form of this theorem can produce even more simple way of checking its validity by finding a_i and a_j in $\{a_1, \ldots, a_n\}$ such that $a_i \neq a_j$ and $\gcd(a_i, a_j) = 1$. This implies $\gcd(a_1, \ldots, a_n) = 1$, then we simply have $\langle a_1, \ldots, a_n \rangle$ is a numerical semigroup.

Definition 2 (Rosales & Garcia-Sanchez, 2009). Let S be a numerical semigroup. We denote the following notations:

1. $G(S) = \mathbb{N}^0 \backslash S$;
2. $g(S) = |G(S)|$;
3. $F(S) = \max\{g : g \in G(S)\}$; and
4. $PF(S) = \{f \in G(S) : f + s \in S$ for all $s \in S \backslash \{0\}\}$.

$G(S)$ is called the set of *gaps* of S, $g(S)$ is called the *genus number* of S, $F(S)$ is the *Frobenius number* of S, and $PF(S)$ is the set of *pseudo-Frobenius numbers* of S.

Definition 3 (Froberg *et al.*, 1987). Let S be a numerical semigroup. The relation $\preccurlyeq_S$ can be defined on $\mathbb{Z}$ by $a \preccurlyeq_S b$, if $b - a \in S$ where $a, b \in \mathbb{Z}$.

Theorem 4 (Froberg *et al.*, 1987). The relation $\preccurlyeq_S$ is the partial order relation on $\mathbb{Z}$.

Theorem 5 (Froberg *et al.*, 1987). Let S be a numerical semigroup and $g \in \mathbb{Z}$. Then $g \in \mathbb{Z} \backslash S$ if and only if $g \preccurlyeq_S f$ for some $f \in PF(S)$.

Based on the work of Froberg et al. (1987) the next remark can be easily derived.

Remark 6. Let S be a numerical semigroup with embedding dimension two or more. If the numerical semigroup S has embedding dimension two, then the set of pseudo-Frobenius *number* must has only one element, which is its Frobenius number. Moreover, the Frobenius number of S must be one of the pseudo-Frobenius *number*.

Theorem 4, Theorem 5, and Remark 6 can be used to simply imply the next Lemma, which is usually used in our research.

Lemma 7. Let S be a numerical semigroup with embedding dimension two. Then, $g \in G(S)$ if and only if $g > 0$ and $F(S) - g \in S$.

Theorem 8. (Johnson, 1960). Let $\{a_1, \ldots, a_n\}$ be the minimal system of generators of a numerical semigroup S. Let $d = gcd\,(a_1, \ldots, a_{n-1})$ and $T = \langle \frac{a_1}{d}, \ldots, \frac{a_{n-1}}{d}, a_n \rangle$. Then

$$F(S) = dF(T) + (d-1)a_n \text{ and } g(S) = dg(T) + \frac{1}{2}(d-1)(a_n - 1).$$

Main Results

From now on, a, b, c are denoted as distinct positive integers such that $a < b < c$.

Lemma 9. Let $\langle a, b\rangle$ be a numerical semigroup. If $g \in G(\langle a, b\rangle)$, then there exists a unique pair $x, y \in \mathbb{N}^0$ such that $ax + by = ab - a - b - g$.

Proof. Let $s \in \langle a, b\rangle$. We first show that if $s < ab$, then there exists a unique pair $x, y \in \mathbb{N}^0$ such that $s = ax + by$. Without loss of generality, we assume $x_1, y_1, x_2, y_2 \in \mathbb{N}^0$ where $x_1 \geq x_2$ be such that $ax_1 + by_1 = s = ax_2 + by_2$ Then, $a(x_1 - x_2) = b(y_2 - y_1) \geq 0$. Since $\langle a, b\rangle$ is a numerical semigroup, we have $\gcd(a, b) = 1$. This implies $a \mid (y_2 - y_1)$ and $b \mid (x_1 - x_2)$. Thus, there exist $l_1, l_2 \in \mathbb{N}^0$ such that $y_2 - y_1 = al_1$ and $x_1 - x_2 = bl_2$. Hence, $y_2 = al_1 + y_1$ and $x_1 = bl_2 + x_2$. This implies that

$$s = a(bl_2 + x_2) + by_1 = abl_2 + (ax_2 + by_1) \geq abl_2.$$

Since $s < ab$, we obtain $l_2 = 0$. Thus, $x_1 = x_2$ and $y_1 = y_2$. Then, S is uniquely determined by x, y. Let $g \in G(\langle a, b\rangle) \subseteq \mathbb{Z}\backslash\langle a, b\rangle$. Since $\{F(\langle a, b\rangle)\} = PF(\langle a, b\rangle)$ and by Theorem 5, we get that $g \preccurlyeq_{\langle a,b\rangle} F(\langle a, b\rangle)$. Then, $F(\langle a, b\rangle) - g \in \langle a, b\rangle$. Hence, there exists $s \in \langle a, b\rangle$ such that $F(\langle a, b\rangle) - g = s$. We see that $s = F(\langle a, b\rangle) - g = (ab - a - b) - g < ab$. Thus, there exists a unique pair $x, y \in \mathbb{N}^0$ such that $s = ax + by$. Thus,

$$ax + by = F(\langle a, b\rangle) - g = ab - a - b - g.$$

It is noticeable that in order to find the solution $x, y \in \mathbb{N}^0$ such that $ax + by = F(\langle a, b\rangle) - g = ab - a - b - g$ where $\langle a, b\rangle$ be a numerical semigroup and $g \in G(\langle a, b\rangle)$. Since $\langle a, b\rangle$ is a numerical semigroup, $\gcd(a, b) = 1$. Thus, there are solutions $x_0, y_0 \in \mathbb{Z}$ such that $ax_0 + by_0 = 1$. Next, we can use the inverse Euclidean algorithm to find a first pair of solution x_0, y_0. Now, since Lemma 9 guarantees the existence of the solution $x, y \in \mathbb{N}^0$, this implies that we can use the linear Diophantine equation to come up with the non-negative solution. Now, for a numerical semigroup S with the minimal system of generators $\{a, b, c\}$. Notice that c cannot be represented as the linear combination of a and b over $\mathbb{N}^0$. We get that $c \in G(\langle a, b\rangle)$. Therefore, Lemma 9 also verifies that the equation

$$ax + by = ab - a - b - c$$

always exists a unique solution on $\mathbb{N}^0$.

Lemma 10. Let $\langle a, b\rangle$ be a numerical semigroup. Then,

$$G(\langle a, b\rangle) = \{F(\langle a, b\rangle) - ax - by > 0 : x, y \in \mathbb{N}^0\}.$$

Proof. Let $g \in G(\langle a, b\rangle)$. By Lemma 7, we have $F(\langle a, b\rangle) - g \in S$. Thus, $F(\langle a, b\rangle) - g = ax + by$ for some $x, y \in \mathbb{N}^0$. Hence, $g = F(\langle a, b\rangle) - ax - by$ for some $x, y \in \mathbb{N}^0$. That is,

$$g \in \{F(\langle a, b\rangle) - ax - by > 0 : x, y \in \mathbb{N}^0\}.$$

Conversely, we let $g \in \{F(\langle a, b\rangle) - ax - by > 0 : x, y \in \mathbb{N}^0\}$. Thus, $g = F(\langle a, b\rangle) - ax_0 - by_0 > 0$ for some $x_0, y_0 \in \mathbb{N}^0$. Thus, $F(\langle a, b\rangle) - g = ax_0 + by_0$ which means $F(\langle a, b\rangle) - g \in S$. Since $g > 0$ and by Lemma 7, we conclude that $g \in G(\langle a, b\rangle)$. Therefore,

$$G(\langle a, b\rangle) = \{F(\langle a, b\rangle) - ax - by > 0 : x, y \in \mathbb{N}^0\}.$$

Theorem 11. Let $\{a, b, c\}$ be the minimal system of generators of S such that $\gcd(a, b) = 1$. If $am + bn = F(\langle a, b\rangle) - c$ for some $m, n \in \mathbb{N}^0$, then

$$g(S) \leq \frac{1}{2}(ab - a - b - 2mn - 2m - 2n - 1).$$

Proof. We define $Q = \{q \in \mathbb{N} : c \preccurlyeq_{\langle a,b\rangle} q \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)\}$. We see that $F(\langle a,b\rangle) - F(\langle a,b\rangle) = 0 \in S$. By Lemma 7 and the transitivity of $\preccurlyeq_{\langle a,b\rangle}$, we get that Q is non-empty and since $c \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)$, we have that Q is also well-defined.

Next, we will show that $G(\langle a,b,c\rangle) \subseteq G(\langle a,b\rangle)\setminus Q$. Let $g \in G(\langle a,b,c\rangle)$. By the definition of $G(\langle a,b\rangle)$, we immediately have that $g \in G(\langle a,b\rangle)$. Now, suppose that $g \in Q$. Hence,

$$c \preccurlyeq_{\langle a,b\rangle} g \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle).$$

Since $c \preccurlyeq_{\langle a,b\rangle} g$, we get $g - c \in \langle a,b\rangle$. That is, there exist $x_0, y_0 \in \mathbb{N}^0$ such that $g - c = ax_0 + by_0$. Notice that $g = ax_0 + by_0 + c \notin G(\langle a,b,c\rangle)$, which is a contradiction. This means that $G(\langle a,b,c\rangle) \subseteq G(\langle a,b\rangle)\setminus Q$. For each $g \in Q$, we have $g \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)$. It follows from Theorem 5 that $g \in G(\langle a,b\rangle)$. Therefore $Q \subseteq G(\langle a,b\rangle)$. Since both $G(\langle a,b\rangle)$ and Q are finite, we obtain that

$$|G(\langle a,b,c\rangle)| \leq |G(\langle a,b\rangle)| - |Q| = \frac{1}{2}(ab - a - b + 1) - |Q|.$$

Now, we are in the position to find the cardinality of Q. By the definition of the partial order $\preccurlyeq_{\langle a,b\rangle}$, we get

$$\begin{aligned} Q &= \{q : c \preccurlyeq_{\langle a,b\rangle} q \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)\} \\ &= \{c + ax + by : x, y \in \mathbb{N}^0 \text{ and } c + ax + by \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)\} \\ &= \{c + ax + by : x, y \in \mathbb{N}^0 \text{ and } a(m - x) + b(n - y) \in \langle a,b\rangle\} \\ &= \{c + ax + by : x = 0,1,\dots,m \text{ and } y = 0,1,\dots,n\} \\ &= \{c + ax + by : (x,y) \in \{0,1,\dots,m\} \times \{0,1,\dots,n\}\}. \end{aligned}$$

Thus, $|Q| = (m+1)(n+1)$. As a result,

$$g(\langle a,b,c\rangle) \leq \frac{1}{2}(ab - a - b + 1) - (m+1)(n+1) = \frac{1}{2}(ab - a - b - 2mn - 2m - 2n - 1).$$

Therefore, we conclude that

$$g(\langle a,b,c\rangle) \leq \frac{1}{2}(ab - a - b - 2mn - 2m - 2n - 1).$$

Theorem 12. Let $\{a,b,c\}$ be the minimal system of generators of $\langle a,b,c\rangle$ such that $\gcd(a,b) = 1$. If $2c, 3c \in \langle a,b\rangle$ and $am + bn = ab - a - b - c$ for some $m, n \in \mathbb{N}^0$, then

$$F(\langle a,b,c\rangle) = ab - a - b - \min\{a(m+1),\ b(n+1)\}.$$

Proof. Since $\gcd(a,b) = 1$, we see that $\langle a,b\rangle$ is a numerical semigroup. Since $\{a,b,c\}$ is the minimal system of generators, we have c cannot be represented as the linear combination of a and b. Thus, $c \in G(\langle a,b\rangle)$. Now, we consider the set $Q = \{q : c \preccurlyeq_{\langle a,b\rangle} q \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)\}$. We will show that $G(\langle a,b,c\rangle) = G(\langle a,b\rangle)\setminus Q$. Assume that $g \in G(\langle a,b\rangle)\setminus Q$ which implies $g \in G(\langle a,b\rangle)$ and $g \notin Q$. By Theorem 5, we have $g \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)$. From $g \notin Q$, we get $g - c \notin \langle a,b\rangle$. Now, we claim that $g \in G(\langle a,b,c\rangle)$.

Suppose that $g \notin G(\langle a,b,c\rangle)$. Then $g \in \langle a,b,c\rangle$. From $g \in G(\langle a,b\rangle)$, we get that g must be in the form $ax + by + ck$ where $x, y, k \in \mathbb{N}^0$ and $k > 0$. There are 2 cases to consider:

Case 1. $k = 1$. Then we have that $g - c = ax_1 + by_1 \in \langle a,b\rangle$, which is a contradiction.

Case 2. $k > 1$. Then $g = ax_1 + by_1 + ck$. We observe that

$$\langle 2c, 3c\rangle = \{0, 2c, 3c, 4c, 5c, \dots\} = \{\alpha c : \alpha \in \mathbb{N}^0 \text{ such that } \alpha \neq 1\}.$$

Hence, we get that $\alpha c \in \langle a,b\rangle$ for all $\alpha \in \mathbb{N}^0\setminus\{1\}$. Thus,

$$g = (ax_1 + by_1) + ck \in G(\langle a,b\rangle), \text{ which is a contradiction.}$$

This implies $g \in G(\langle a,b,c\rangle)$. That is,

$$G(\langle a,b\rangle)\backslash Q \subseteq G(\langle a,b,c\rangle).$$

Also, we note from the proof of Theorem 11 that $G(\langle a,b,c\rangle) \subseteq G(\langle a,b\rangle)\backslash Q$. Thus, we conclude that

$$G(\langle a,b,c\rangle) = G(\langle a,b\rangle)\backslash Q.$$

Next, we will prove that $F(\langle a,b\rangle) = ab - a - b - \min\{a(m+1), b(n+1)\}$. By assumption, we obtain that $c = F(\langle a,b\rangle) - am - bn$. From the proof of Theorem 11, we get that

$$\begin{aligned} Q &= \{c + ax + by : x,y \in \mathbb{N}^0 \text{ and } a(m-x) + b(n-y) \in \langle a,b\rangle\} \\ &= \{F(\langle a,b\rangle) - a(m-i) - b(n-j) : i,j \in \mathbb{N}^0 \text{ and } a(m-i) + b(n-j) \in \langle a,b\rangle\} \\ &= \{F(\langle a,b\rangle) - a(m-i)b(-j) : i = 0,1,\dots,m \text{ and } j = 0,1,\dots,n\} \\ &= \{F(\langle a,b\rangle) - ai - bj : i = 0,1,\dots,m \text{ and } j = 0,1,\dots,n\} \\ &= \{F(\langle a,b\rangle) - ai - bj : (i,j) \in \{0,1,\dots,m\} \times \{0,1,\dots,n\}\}. \end{aligned}$$

It follows from Lemma 9 that for each $g \in G(\langle a,b\rangle)$, there exists a unique pair $\alpha,\beta \in \mathbb{N}^0$ such that

$$g = ab - a - b - a\alpha - b\beta = F(\langle a,b\rangle) - a\alpha - b\beta.$$

Then by Lemma 10, we can specify all elements in $G(\langle a,b\rangle)$ as

$$G(\langle a,b\rangle) = \{F(\langle a,b\rangle) - a\alpha - b\beta \in \mathbb{N} : \alpha,\beta \in \mathbb{N}^0\}.$$

Now, we let m^* be the greatest number such that

$$F(\langle a,b\rangle) - am^* > 0.$$

Thus,

$$\begin{aligned} G(\langle a,b\rangle)\backslash Q &= \{F(\langle a,b\rangle) - a\alpha - b\beta \in \mathbb{N} : \alpha,\beta \in \mathbb{N}^0 \text{ such that } g \notin Q\} \\ &= \{F(\langle a,b\rangle) - a\alpha - b\beta \in \mathbb{N} : \alpha,\beta \in \mathbb{N}^0 \text{ such that } (\alpha,\beta) \notin \{0,\dots,m\} \times \{0,\dots,n\}\} \\ &= \cup_{i=0}^{m}\{F(\langle a,b\rangle - ai - b\beta \in \mathbb{N} : \beta \in \mathbb{N}^0 \text{ such that } \beta > n\} \\ &\quad \cup \cup_{i=m+1}^{m^*}\{F(\langle a,b\rangle - ai - b\beta \in \mathbb{N} : \beta \in \mathbb{N}^0\}. \end{aligned}$$

We have that

$$\begin{aligned} F(\langle a,b,c\rangle) &= \max\{g : g \in G(\langle a,b,c\rangle)\} \\ &= \max\{g : g \in G(\langle a,b\rangle)\backslash Q\} \\ &= \max\left(\cup_{i=0}^{m}\{F(\langle a,b\rangle - ai - b\beta \in \mathbb{N} : \beta \in \mathbb{N}^0 \text{ such that } \beta > n\}\right. \\ &\quad \left. \cup \cup_{i=m+1}^{m^*}\{F(\langle a,b\rangle - ai - b\beta \in \mathbb{N} : \beta \in \mathbb{N}^0\}\right) \\ &= F(\langle a,b\rangle) - \min\left(\cup_{i=0}^{m}\{ai + b\beta \in \mathbb{N} : \beta \in \mathbb{N}^0 \text{ such that } \beta > n\}\right. \\ &\quad \left. \cup \cup_{i=m+1}^{m^*}\{ai + b\beta : \beta \in \mathbb{N}^0\}\right) \\ &= F(\langle a,b\rangle) - \min(\{b\beta : \beta \in \mathbb{N}^0 \text{ such that } \beta > n\} \cup \{ai : i = m+1, m+2, m+3, \dots, m^*\}) \\ &= F(\langle a,b\rangle) - \min\{a(m+1), b(n+1)\}\text{, and we are done.} \end{aligned}$$

From now on, we define the set Q depends on the numerical semigroup $\langle a,b,c\rangle$ that is minimally generated by $\{a,b,c\}$ by

$$Q = \{q : c \preccurlyeq_{\langle a,b\rangle} q \preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)\}.$$

As we proceed the proofs, we have that the other representation of Q here,

$$\begin{aligned} Q &= \{c + ax + by : (x,y) \in \{0,1,\dots,m\} \times \{0,1,\dots,n\}\} \\ &= \{F(\langle a,b\rangle) - ai - bj : (i,j) \in \{0,1,\dots,m\} \times \{0,1,\dots,n\}\} \end{aligned}$$

where $am + bn = ab - a - b - c$. We observe that $\langle a,b\rangle \cup Q \subseteq \langle a,b,c\rangle$ and $\langle a,b\rangle \cap Q = \emptyset$.

The following results are simply obtained by applying Theorem 11 and Theorem 12. We note here that the next theorem is the upper bound of the Frobenius number of the numerical semigroup such that it is minimally generated by $\{a,b,c\}$ where $\gcd(a,b)=1$.

Theorem 13 Let $\{a,b,c\}$ be the minimal system of generators of $\langle a,b,c\rangle$ such that $\gcd(a,b)=1$. If $am+bn=ab-a-b-c$ for some $m,n\in\mathbb{N}^0$, then

$$F(\langle a,b,c\rangle)\leq ab-a-b-\min\{a(m+1),b(n+1)\}.$$

Proof. We let $Q=\{q : c\preccurlyeq_{\langle a,b\rangle} q\preccurlyeq_{\langle a,b\rangle} F(\langle a,b\rangle)\}$. By the proof of Theorem 11 and Theorem 12, we have $G(\langle a,b,c\rangle)\subseteq G(\langle a,b\rangle)\backslash Q$ and $\max(G(\langle a,b\rangle)\backslash Q)=F(\langle a,b\rangle)-\min\{a(m+1),b(n+1)\}$ respectively. Therefore,

$$\begin{aligned}F(\langle a,b,c\rangle)&=\max(G(\langle a,b,c\rangle))\\&\leq\max(G(\langle a,b\rangle)\backslash Q)\\&=F(\langle a,b\rangle)-\min\{a(m+1),b(n+1)\}.\end{aligned}$$

Corollary 13. Let $\{a,b,c\}$ be the minimal system of generators of $\langle a,b,c\rangle$ such that $\gcd(a,b)=1$. If $2c,3c\in\langle a,b\rangle$ and $am+bn=ab-a-b-c$ for some $m,n\in\mathbb{N}^0$, then

$$g(\langle a,b,c\rangle)=\frac{1}{2}(ab-a-b-2mn-2m-2n-1).$$

Proof. By the proof of Theorem 12, we have

$$G(\langle a,b,c\rangle)=G(\langle a,b\rangle)\backslash Q.$$

Since both $G(\langle a,b\rangle)$ and Q are finite, we obtain that

$$|G(\langle a,b,c\rangle)|=|G(\langle a,b\rangle)|-|Q|=\frac{1}{2}(ab-a-b+1)-|Q|.$$

By the proof of Theorem 10, we have

$$|Q|=(m+1)(n+1).$$

Thus,
$$\begin{aligned}|G(\langle a,b,c\rangle)|&=\frac{1}{2}(ab-a-b+1)-(m+1)(n+1)\\&=\frac{1}{2}(ab-a-b-2mn-2m-2n-1).\end{aligned}$$

Now, we are in the position to extend our previous results for the case $\gcd(a,b)\geq 1$. It is clearly seen that all formulas that are previously obtained for a given numerical semigroup $\langle a,b,c\rangle$ are applicable whenever $\gcd(a,b)=1$. However, if $\gcd(a,b)\geq 1$, we can apply Theorem 8 to make our formulas can be computed more generally. The next corollary is our extension form of our formulas for the case $d\geq 1$.

Corollary 14. Let $\{a,b,c\}$ be the minimal generators of a numerical semigroup $\langle a,b,c\rangle$ such that $gcd\ (a,b)=d$. If $\frac{a}{d}m+\frac{b}{d}n=\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-c$ for some $m,n\in\mathbb{N}^0$, and $\{\frac{a}{d},\frac{b}{d},c\}$ is the minimal system of generators of $\langle\frac{a}{d},\frac{b}{d},c\rangle$, then

$$F(\langle a,b,c\rangle)\leq\frac{ab}{d}-a-b+(d-1)c-\min\{a(m+1),b(n+1)\};\text{ and}$$
$$g(\langle a,b,c\rangle)\leq\frac{1}{2d}(ab-ad-bd-2mnd^2-2md^2-2nd^2-(2d-1)d+(d-1)cd).$$

Moreover, if $2c,3c\in\langle\frac{a}{d},\frac{b}{d}\rangle$, then

$$F(\langle a,b,c\rangle)=\frac{ab}{d}-a-b+(d-1)c-\min\{a(m+1),b(n+1)\};\text{ and}$$
$$g(\langle a,b,c\rangle)=\frac{1}{2d}(ab-ad-bd-2mnd^2-2md^2-2nd^2-(2d-1)d+(d-1)cd).$$

Proof. Let $\langle a,b,c\rangle$ be a numerical semigroup, and $d=\gcd(a,b)$. Then, $\gcd(\frac{a}{d},\frac{b}{d})=1$. Thus, by Theorem 11 and Theorem 13, we have

$$F\left(\left\langle\frac{a}{d},\frac{b}{d},c\right\rangle\right)\leq\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-\min\{\frac{a}{d}(m+1),\frac{b}{d}(n+1)\};\text{ and}$$

$$g\left(\left\langle\frac{a}{d},\frac{b}{d},c\right\rangle\right) \le \frac{1}{2}\left(\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-2mn-2m-2n-1\right).$$

Hence, by Theorem 8, we obtain

$$g(\langle a,b,c\rangle) \le \frac{d}{2}\left(\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-2mn-2m-2n-1\right)+\frac{1}{2}(d-1)(c-1)$$
$$=\frac{1}{2d}(ab-ad-bd-2mnd^2-2md^2-2nd^2-(2d-1)d+(d-1)cd).$$

Also, $$F(\langle a,b,c\rangle) \le d\left(\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-\min\{\frac{a}{d}(m+1),\frac{b}{d}(n+1)\}\right)+(d-1)c$$
$$=\frac{ab}{d}-a-b+(d-1)c-\min\{a(m+1),b(n+1)\}.$$

Now, if $2c,3c \in \langle\frac{a}{d},\frac{b}{d}\rangle$, then we apply Theorem 12 and Corollary 13. Thus,

$$F\left(\left\langle\frac{a}{d},\frac{b}{d},c\right\rangle\right)=\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-\min\{\frac{a}{d}(m+1),\frac{b}{d}(n+1)\};\text{ and}$$
$$g\left(\left\langle\frac{a}{d},\frac{b}{d},c\right\rangle\right)=\frac{1}{2}\left(\frac{a}{d}\frac{b}{d}-\frac{a}{d}-\frac{b}{d}-2mn-2m-2n-1\right).$$

Hence, by Theorem 8, we have that

$$g(\langle a,b,c\rangle)=\frac{1}{2d}(ab-ad-bd-2mnd^2-2md^2-2nd^2-(2d-1)d+(d-1)cd)$$
$$F(\langle a,b,c\rangle)=\frac{ab}{d}-a-b+(d-1)c-\min\{a(m+1),\ b(n+1)\}.$$

Example 15 Consider a numerical semigroup $\langle 12,14,23\rangle$.

Observe that $\{12,14,23\}$ is a minimal system of generator. From $\gcd(12,14)=2$, we observe that $\{6,7,23\}=\{\frac{12}{2},\frac{14}{2},23\}$ is the minimal system of generator, and

$$F(\langle 6,7,23\rangle)=F\left(\left\langle\frac{12}{2},\frac{14}{2},23\right\rangle\right)=(6)(7)-23=19.$$

Since $46=2(23)>19$ and $69=3(23)>19$, $2(23)\in\langle 6,8\rangle$ and $3(23)\in\langle 6,8\rangle$. Thus, we can use Corollary 14. Now, consider the equation,

$$\frac{12}{2}m+\frac{14}{2}n=\frac{12}{2}\frac{14}{2}-\frac{12}{2}-\frac{14}{2}-23$$
$$6m+7n=42-6-7-23=6.$$

Note that Lemma 9 guarantees the existence of the solution $m,n\in\mathbb{N}^0$. Clearly, that $m=1,n=0$. Thus,

$F(\langle 12,14,23\rangle)=\frac{(12)(14)}{2}-12-14+(2-1)(23)-\min\{12(1+1),14(0+1)\}=67$. Also, $g(\langle 12,14,23\rangle)=\frac{1}{4}(168-24-28-8-6+46)=37$.

Discussion

Based on techniques used in this research to find the Frobenius number in three variables, the partial order relation is formed on the numerical semigroup with embedding dimension two. In detail, to form this partial order relation, the property of the pseudo-Frobenius number, which has Frobenius number as its element, is used mainly to provide the way of finding *gaps* and applied to find the largest gap, which is the Frobenius number. In addition, for a numerical semigroup generated by $\langle a,b,c\rangle$, the formula can be used to calculate to exact formula result of the Frobenius number and genus number whenever $2c,3c\in\langle a,b\rangle$. However, if the condition $2c,3c\in\langle a,b\rangle$ is not provided, it is guaranteed to yield the upper bound of the result in general cases. To extend the same technique for the formula of finding the Frobenius number in three variables, one can consider the case of c as followings: $2c\in\langle a,b\rangle, 3c\notin\langle a,b\rangle$ and $2c\notin\langle a,b\rangle, 3c\in\langle a,b\rangle$.

However, to find the Frobenius number using the same techniques for four our more variables, it is not guarantee for a numerical semigroup with embedding dimension more than two that the set of pseudo-Frobenius number must have only one element. For example, consider a numerical semigroup $\langle 5,6,7\rangle$. It can be seen that the set of pseudo-Frobenius number is $\{8,9\}$. In other cases, it also possible that the given numerical semigroup with embedding dimension more than two might have the only one pseudo-Frobenius number, such as $\langle 5,12,18\rangle$. This numerical semigroup has $\{31\}$. Thus, this is possible to use the same technique to find the formula of the Frobenius number where the set of pseudo-Frobenius number is guaranteed to have only one element.

Conclusion and Suggestions

This work is done using a numerical semigroup concept of forming the partial order relation based on the only number of elements in the set of the set of pseudo-Frobenius number. For a numerical semigroup S with the minimal system of generator $\{a,b,c\}$, first we provide the exact formula of calculation the Frobenius and genus number for the case $2c,3c \in \langle a,b\rangle$ where $\gcd(a,b)=1$. However, if the condition $2c,3c \in \langle a,b\rangle$ is not provided, the result is guaranteed to yield the upper bound of the Frobenius and genus number in general cases. Next, Theorem 8 is then used to extend the result for the case $\gcd(a,b)=d$. Again, the exact result is only guaranteed for the case $2c,3c \in \langle \frac{a}{d},\frac{b}{d}\rangle$ otherwise the result will provide the upper bound of the Frobenius and genus number.

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