Performance of Seven Statistics for Mean Difference Testing Between Two Populations Under Combined Assumption Violations

Montri Sangthong and Praphat Klubnual*

Division of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Suvarnabhumi, Ayutthaya, 13000 Thailand

* Corresponding author. E-mail address: praphat.k@rmutsb.ac.th

Received: 3 August 2020; Revised: 19 February 2021; Accepted: 25 February 2021; Available online: 6 May 2021

Abstract

The objective of this research was to compare the performance of seven statistics for mean difference testing between two populations when data did not follow assumptions whereas the simulation conditions were determined as 5 distributions, variance, and sample size in both cases are equal and unequal. The results showed that when the population had log-normal distribution, gamma distribution and poisson distribution and equal variance, the Welch Based on Rank test (WBR test) were most effective. When the population had log-normal distribution, gamma distribution, exponential distribution, poisson distribution and uniform distribution and unequal variance, the Welch t test was distinctively found to have a higher performance than others testing statistics.

Keywords: Two-sample location test, Parametric test, Non-parametric test, Non-parametric bootstrap test, t test, Welch t test, Welch Based on Rank test, Brunner-Munzel test, Yuen-Welch test, Exact Wilcoxon signed-rank test

Introduction

The testing of differences on location or central tendency values between two independent populations is widely used for hypothesis testing in several research fields, such as educational research, medicine, business administration, etc. The said hypothesis testing normally applies an independent t test when both populations have a normal distribution and equal variances. However, if both populations have a normal distribution and unequal variances, the Welch t test is used. These statistics are in the parametric test (Stonehouse & Forrester, 1998; Fagerland & Sandvik, 2009; Ahad, Othman, & Yahaya, 2011). However, if the populations have a non-normal distribution, the hypothesis test will be performed using nonparametric statistics. The most popular testing statistic is Wilcoxon-Mann-Whitney (Mann, & Whitney, 1947; Harwell & Serlin, 1989; Fagerland & Sandvik, 2009). It could be said that the said testing statistics, in many conditions, experience problems with efficiency, especially when the populations have an unequal variance (Harwell, Rubinstein, Hayes, & Olds, 1992; Zimmerman & Zumbo, 1993a, 1993b; Stonehouse & Forrester, 1998). Afterwards, the testing statistics are adjusted to what is called Brunner-Munzel (BM). It aims to present the distinction when there are unequal variances, known as the Behrences-Fisher problem (Brunner & Munzel, 2000). Nevertheless, there is an argument about the effectiveness of the above testing statistics (Medina, Kimberg, Chatterjee, & Coslett, 2010). Also, the Welch t test is used for hypothesis testing by transforming data into ranking order (Welch, 1938; Reiczigel, Zakarias, & Rozsa, 2005; Winter, 2013).

In addition, testing statistics are developed to test the mean difference when the population experiences a skewed distribution or outlier value by using the trimmed means. In detail, before finding the mean, the data is in ascending order and the smallest and highest values are cut out. It could be said that the maximum efficiency is found by cutting out 20 percent of the data (Wilcox, 1994; Wicox, 2005). This comparison of trimmed means is called the Yuen-welch test statistics (Yuen, 1974). The study of Fagerland and Sandvik (2009) revealed that the above test statistics are suitable for analysis with a positive skewed distribution. For hypothesis testing relating the mean difference, the standard error could be adjusted by using the non-parametric bootstrap method, in which the algorithm was proposed by Efron & Tibshirani (1993). The outstanding feature of this method is that there is no assumption on the normal distribution and the equality of variance because it is done by replacing the sampling for estimating the parameters.

In many research fields, data sometimes may not meet the assumptions on normal distribution and they mostly occur with unequal variances (Wilcox, 1990; Wilcox & Keselman, 2003). For instance, for research in medical fields, data are normally skewed (Bridge & Sawilowsky, 1999). If two skewed distribution have unequal locations, it can be expected that the variances will also differ. Later on, medical data regularly show an integration of skewness and unequal variances (Fagerland & Sandvik, 2009). Therefore, data analysis using parametric statistics is not proper. The objective of this research is to compare the efficiency of the test statistic both parametric statistics and non-parametric statistics, such as the Welch Based on Rank test (WBR test), Yuen-Welch test (Y-W test), Brunner-Munzel test (B-M test) and Exact Wilcoxon signed-rank test. The test statistic is done by considering the efficiency of capacity to control type I errors and testing power. The capacity to control type I errors is based on Bradley (1978) criteria by testing the two-tail hypothesis at a significant level of 0.05 and 0.01. If the values range between 0.025-0.075 or 0.005-0.015 respectively, it is considered to have the capacity to control a type I error.

Methods and Materials

Statistical Method

The test statistic for the hypothesis test on different locations of two independent populations were as follows:

1. Independent sample t test (t test)

The independent sample t test (t test) was used for parametric statistics with assumptions consisting of normal distribution and equal variances population. The details are: (Fagerland & Sandvik, 2009)

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_P^2}{n_1} + \frac{S_P^2}{n_2}}} \qquad (1)$$

Where t is t-distribution

$$S_P^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

The reject H_0 is $t \le -t_{\alpha/2, n_1+n_2-1}$ or $t \ge t_{\alpha/2, n_1+n_2-1}$

2. Welch t test

The Welch t test was used for parametric statistics with the assumption of normal distribution. The Welch t test was proposed by Welch (1937) for unequal variances. The details are:

$$\text{Welch} = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \quad (2)$$

Where Welch is t-distribution

$$S_1^2 = \frac{\sum_{i=1}^{n_1}\left(X_{i1} - \bar{X}_1\right)^2}{n_1 - 1}$$

$$S_2^2 = \frac{\sum_{i=1}^{n_1}\left(X_{i2} - \bar{X}_2\right)^2}{n_2 - 1}$$

The reject H_0 is Welch $\leq -t_{\alpha/2,df}$ or Welch $\geq t_{\alpha/2,df}$ by $df = \dfrac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(S_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(S_2^2/n_2\right)^2}{n_2 - 1}}$

3. t-test based on bootstrap standard errors (t-BSE test)

Hypothesis testing with t-BSE was described as follows (Efron & Tibshirani, 1993):

1) Evaluated $\hat{\theta} = \bar{X}_1 - \bar{X}_2$

2) Evaluated the standard error by replacement sampling at sample sizes n_1^* and n_2^* from $n_1 + n_2$ and evaluated $\hat{\theta}^*(b) = \bar{X}_1^* - \bar{X}_2^*$

3)Evaluated $\hat{se}_B = \left\{\sum_{b=1}^{B}\left[\hat{\theta}^*(b) - \hat{\theta}^*(.)\right]^2 / B - 1\right\}^{1/2}$, for this paper, Bootstrap was 1,000 time, where $\hat{\theta}^*(.) = \dfrac{\sum_{b=1}^{B}\hat{\theta}^*(b)}{B}$

4) Evaluated $t - BSE = \dfrac{\hat{\theta}}{\hat{se}_B}$ (3)

5) Rejected H_0 when $t - BSE \leq -t_{\alpha/2,B-1}$ or $t - BSE \geq t_{\alpha/2,B-1}$

4. Welch Based on Rank test (WBR test)

Hypothesis testing with Welch Based on Rank was described as follows: (Welch, 1938; Reiczigel, et al., 2005; Winter, 2013).

1) Combined n_1 and n_2 and ascended the data, if tied values was found, the midranks were averaged.

2) Calculated the mean and standard deviation of midrank in the first and second samples as $\bar{R}_1$ and $\bar{R}_2$, and calculated S_1^2 and S_2^2 respectively

3)Evaluated the test statistic $WBR = \dfrac{\bar{R}_1 - \bar{R}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$ (4)

4) Rejected H_0 when $WBR \le -t_{\alpha/2,df}$ or $WBR \ge t_{\alpha/2,df}$ by $df = \dfrac{\left(\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(S_1^2/n_1\right)^2}{n_1-1}+\dfrac{\left(S_2^2/n_2\right)^2}{n_2-1}}$

5. Brunner-Munzel test (B-M test)

Brunner and Munzel (2000) proposed the hypothesis testing method to solve the Behrens-Fisher problem or unequal variance. Following the notation R_1 and R_2 are the midranks of n_1 and n_2 based on pooling all observations, V_1 and V_2 were the midranks within each sample. $\bar{R}_1$ and $\bar{R}_2$ were the means of the pooled midranks. The test statistics were detailed as follows:

$$\text{B-M} = \frac{\bar{R}_1-\bar{R}_2}{\left(n_1+n_2\right)\sqrt{\dfrac{SB_1^2}{n_1 n_2^2}+\dfrac{SB_2^2}{n_1^2 n_2}}} \quad (5)$$

where $$SB_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}\left(R_1^i - V_1^i - \bar{R}_1 + \frac{n_1+1}{2}\right)^2$$

$$SB_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}\left(R_2^i - V_2^i - \bar{R}_2 + \frac{n_2+1}{2}\right)^2$$

Reject H_0 when B-M $\le -t_{\alpha/2,df}$ or B-M $\ge t_{\alpha/2,df}$ by $$df = \frac{\left(\dfrac{SB_1^2}{n_1}+\dfrac{SB_2^2}{n_2}\right)^2}{\dfrac{SB_1^4}{n_1^2\left(n_2-1\right)}+\dfrac{SB_2^4}{n_2^2\left(n_1-1\right)}}$$

6. Yuen-Welch test (Y-W test)

Yuen-Welch test used trimmed means by cutting of 20% or $\gamma = 0.2$, which is the best option (Wilcox, 1994; Wilcox, 2005). Specifically, the least and greatest values were reduced by 20% and defined $g_1 = \gamma n_1$ and $g_2 = \gamma n_2$ as trimmed observations from each tail in n_1 and n_2. Indicated the number of remaining observations in the trimmed samples by $h_1 = n_1 - 2g_1$ and $h_2 = n_2 - 2g_2$. SW_1^2 and SW_1^2 were sample winsorized variances in each sample. This test statistic could be explained as follows (Yuen, 1974):

$$\text{Y-W} = \frac{\bar{X}_{\gamma 1}-\bar{X}_{\gamma 2}}{\sqrt{d_1+d_2}} \quad (6)$$

where $$d_1 = \frac{SW_1^2\left(n_1-1\right)}{h_1\left(h_1-1\right)}$$

$$d_2 = \frac{SW_2^2\left(n_2-1\right)}{h_2\left(h_2-1\right)}$$

$$SW_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}\left(W_1^i-\bar{W}_1\right)^2$$

$$SW_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}\left(W_2^i-\bar{W}_2\right)^2$$

Rejected H_0 when $Y-W \le -t_{\alpha/2,df}$ or $Y-W \ge t_{\alpha/2,df}$ by $df = \dfrac{(d_1+d_2)^2}{\left(\dfrac{d_1^2}{h_1-1}+\dfrac{d_2^2}{h_2-1}\right)}$

7) Exact Wilcoxon signed-rank test

The Exact Wilcoxon signed-rank test was proposed by Wilcoxon (1945). The Exact Wilcoxon signed-rank test can apply the exact permutation distribution of ranks since the exact test is only practicable for small sample. However, due to the current advances in computers, P-value is calculated from the actual distribution except when there are duplicates (tied) and approximated by normal distribution. For computation of such test statistics, WRS2 in R program command is used.

The Exact Wilcoxon signed-rank test were designed based on W rank statistics used for ordering the samples which are the combination of $N = n_1 + n_2$ X_1 -value and X_2 -value ranking from least to highest determining S_1 as the rank of $X_{21},...,X_{2n_2}$ as the rank of X_{2n_2} in this joint ordering. That is, W refers to the sum of the ranks determined for the X_2 -value.

$$W = \sum_{j=1}^{n_2} S_j. \tag{7}$$

Reject H_0 when $W \le n_2(n_1+n_2+1) - W_{\alpha/2}$ or $W \ge W_{\alpha/2}$

Methodology

The Monte Carlo simulation was used to estimate the type I error and the power of the test. R programVersion 3.5.2 was written to generate data for the following conditions. In each case, 10,000 iterations of data sets were generated.

1. Generating of two population data groups into five distributions.

1) Population with log-normal distribution

$$X_1 \sim \log \text{Norm}(\mu=0,\sigma=1)-1.65 \tag{8}$$

$$X_2 \sim \sigma_2(\log \text{Norm}(\mu=0,\sigma=1)-1.65-\text{shift}) \tag{9}$$

2) Population with gamma distribution

$$X_1 \sim \text{gamma}(\alpha=1,\beta=1)-1 \tag{10}$$

$$X_2 \sim \sigma_2(\text{gamma}(\alpha=1,\beta=1)-1-\text{shift}) \tag{11}$$

3) Population with exponential distribution

$$X_1 \sim \exp(\lambda=0.5)-0.5 \tag{12}$$

$$X_2 \sim \sigma_2(\exp(\lambda=0.5)-0.5-\text{shift}) \tag{13}$$

4) Population with poission distribution

$$X_1 \sim \text{pois}(\lambda=5)-5 \tag{14}$$

$$X_2 \sim \sigma_2(\text{pois}(\lambda=5)-5-\text{shift}) \tag{15}$$

5) Population with uniform distribution

$$X_1 \sim \text{unif}(\min=0,\max=1)-0.5 \tag{16}$$

$$X_2 \sim \sigma_2(\text{unif}(\min = 0, \max = 1) - 0.5 - \text{shift}) \quad (17)$$

When determine shift = 0 as the calculation for type I error and shift = 1, 1, 0.5, 1 and 0.25 when population had log-normal distribution, gamma distribution, exponential distribution, poission distributionand uniform distribution, respectively, was the calculation of power test.

2. Determining four variance ratios for both equal and unequal variance, which were 1:1, 1:2.25, 1:4, and 1:9.

3. Determiningthe sample sizes for both equal and unequal sizes consisting of seven sizes, which were (10,10), (10,30), (30,10), (30,30), (50,100), (100,50), and (100,100).

4. Finding the probability of the type I error by assigning the equivalent mean ratio and counting the rejection number of H_0, then dividing it by the 10,000 iteration rounds for each condition. This hypothesis testing was done at a significant level of 0.05 and 0.01. If the probability of a type I error was between 0.025-0.075or0.005-0.015 respectively, it would have been considered to have the ability to control the type I error based on criteria from Bradley (1978).

5. Finding the testing power by assigning the difference of the mean based on the effect size. If it is above 0.80, it is considered to have a high effect (Cohen, 1988). The testing power was calculated by counting the rejection number of H_0 and dividing it by 10,000. This hypothesis testing was done at a significant level of 0.05 and 0.01.

6. Analyzing the performance of each situation test statistics that can control type I error and has the highest Power of the test

Results

The results of simulation were reported in following order in type I errors and power of the test for testing statistics had the following details:

Table 1 The probability of a type I error and power test (0.05 significant level) of statistics on location testing between two populations with log-normal distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0351***	0.3092	***0.0274***	0.2960	***0.0383***	0.3174	***0.0551***	**0.5339**	***0.0579***	0.4983	***0.0302***	0.3860	***0.0436***	0.5026
	1:2.25	***0.0517***	0.4304	***0.0460***	0.4200	***0.0559***	0.4387	0.1174	0.6589	0.1103	0.5809	***0.0692***	**0.5873**	0.1040	0.6361
	1:4	0.0855	0.5037	0.0802	0.4908	0.0889	0.5093	0.1932	0.7127	0.1676	0.6106	0.1400	0.6713	0.1755	0.6969
	1:9	0.1292	0.5460	0.1221	0.5308	0.1328	0.5514	0.2552	0.7385	0.1982	0.5976	0.2202	0.7132	0.2404	0.7271
10:30	1:1	***0.0409***	0.3634	***0.0697***	0.2783	***0.0428***	0.3661	**0.0507**	**0.8950**	***0.0549***	0.8815	***0.0535***	0.5362	***0.0457***	0.7738
	1:2.25	***0.0364***	0.4207	***0.0382***	0.5105	***0.0370***	0.4256	0.2027	0.9809	0.1965	0.9660	***0.0528***	**0.8814**	0.1109	0.9061
	1:4	***0.0351***	0.4417	***0.0375***	**0.6101**	***0.0353***	0.4448	0.3985	0.9880	0.3558	0.9739	0.1416	0.9461	0.2015	0.9314
	1:9	***0.0315***	0.4575	***0.0572***	**0.6785**	***0.0329***	0.4599	0.5778	0.9941	0.4832	0.9844	0.3003	0.9789	0.3071	0.9538
30:10	1:1	***0.0439***	0.3784	***0.0687***	0.4685	***0.0452***	0.3820	***0.0535***	0.5704	***0.0570***	0.5517	***0.0551***	0.5456	***0.0505***	**0.7003**
	1:2.25	0.0920	0.5714	0.1150	0.5149	0.0946	0.5747	0.1264	0.6607	0.1227	0.6183	0.1355	0.6406	0.2029	0.8084
	1:4	0.1515	0.6497	0.1282	0.5400	0.1550	0.6512	0.1811	0.6947	0.1659	0.6411	0.1814	0.6794	0.2987	0.8351
	1:9	0.2395	0.7011	0.1410	0.5435	0.2444	0.7027	0.2363	0.7227	0.2059	0.6404	0.2301	0.7125	0.3839	0.8556
30:30	1:1	***0.0379***	0.5474	***0.0365***	0.5453	***0.0393***	0.5505	***0.0468***	**9511.0**	***0.0480***	0.9411	***0.0384***	0.8395	***0.0468***	**0.9511**
	1:2.25	***0.0554***	0.6599	***0.0537***	0.6566	***0.0564***	**0.6615**	0.2799	0.9868	0.2640	0.9782	0.1293	0.9435	0.2799	0.9868
	1:4	***0.0690***	0.6965	***0.0675***	0.6925	***0.0708***	**0.6980**	0.4307	0.9909	0.3852	0.9848	0.2192	0.9685	0.4310	0.9910
	1:9	0.0953	0.7136	0.0939	0.7083	0.0972	0.7154	0.5832	0.9929	0.5078	0.9866	0.3558	0.9801	0.5848	0.9932
50:100	1:1	***0.0448***	0.7897	***0.0519***	0.8279	***0.0462***	0.7908	***0.0493***	**1.0000**	***0.0503***	**1.0000**	***0.0466***	0.9982	***0.0476***	**1.0000**
	1:2.25	***0.0296***	0.8496	***0.0434***	**0.9144**	***0.0290***	0.8491	0.5947	1.0000	0.5705	1.0000	0.1797	1.0000	0.4967	1.0000
	1:4	***0.0307***	0.8539	***0.0479***	**0.9247**	***0.0326***	0.8547	0.8878	1.0000	0.8554	1.0000	0.4562	1.0000	0.8120	1.0000
	1:9	***0.0302***	0.8779	***0.0618***	**0.9460**	***0.0312***	0.8788	0.9714	1.0000	0.9509	1.0000	0.7324	1.0000	0.9330	1.0000
100:50	1:1	***0.0467***	0.7920	***0.0446***	0.7704	***0.0467***	0.7925	***0.0494***	0.9978	***0.0500***	0.9975	***0.0470***	0.9793	***0.0497***	**0.9990**
	1:2.25	0.0820	0.8589	***0.0656***	**0.7999**	0.0832	0.8581	0.4281	0.9999	0.4116	0.9998	0.1859	0.9966	0.5049	1.0000
	1:4	0.1190	0.8747	0.0802	0.8049	0.1195	0.8751	0.6550	1.0000	0.6257	1.0000	0.3346	0.9991	0.7406	1.0000
	1:9	0.1653	0.9062	0.0970	0.8351	0.1658	0.9063	0.7962	1.0000	0.7578	1.0000	0.5166	0.9994	0.8705	1.0000

Table 1 (Cont.)

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
100:100	1:1	***0.0489***	0.9055	***0.0485***	0.9051	***0.0492***	0.9051	***0.0541***	**1.0000**	***0.0542***	**1.0000**	***0.0493***	0.9999	***0.0540***	**1.0000**
	1:2.25	***0.0552***	0.9462	***0.0551***	0.9457	***0.0556***	**0.9463**	0.6843	1.0000	0.6601	1.0000	0.2612	1.0000	0.6836	1.0000
	1:4	***0.0650***	**0.9365**	***0.0646***	0.9355	***0.0650***	0.9361	0.9042	1.0000	0.8811	1.0000	0.5095	1.0000	0.9042	1.0000
	1:9	***0.0737***	**0.9383**	***0.0724***	0.9372	***0.0732***	0.9378	0.9718	1.0000	0.9549	1.0000	0.7361	1.0000	0.9718	1.0000

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 1, it could be found that when the populations had log-normal distribution and equal variance at 0.05 significant level, all test statistics could control type I errors in all sample sizes. For unequal variance cases Welch t test,t test and t-BSE test could control type I errors in almost all sample sizes, unless small sample size and higher variance another group.

It could be found that when the population had log-normal distribution and equal variance, the test statistics with the highest testing power that can control type I errors are WBR test (71.43%) and Exact Wilcoxon signed-rank test (71.43%). For unequal variance, the test statistics with the highest testing power that could control were Welch t test (28.57%), t-BSE test (14.29%)and t test (9.52%) respectively.

Table 2 The probability of a type I error and power test (0.01 significant level) of statistics on location testing between two populations with log-normal distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0051***	0.1395	0.0031	0.1214	0.0038	0.1222	***0.0108***	**0.3031**	0.0159	0.3099	0.0047	0.1572	***0.0078***	0.2740
	1:2.25	***0.0101***	**0.2664**	***0.0076***	0.2487	***0.0076***	0.2457	0.0373	0.4121	0.0447	0.3721	0.0168	0.3624	0.0323	0.3947
	1:4	0.0269	0.3467	0.0243	0.3287	0.0227	0.3251	0.0729	0.4645	0.0713	0.4017	0.0484	0.4634	0.0639	0.4535
	1:9	0.0587	0.4085	0.0538	0.3794	0.0518	0.3896	0.1099	0.4883	0.0968	0.4097	0.1045	0.5212	0.1036	0.4819
10:30	1:1	***0.0090***	0.1908	0.0179	0.0765	***0.0077***	0.1820	***0.0139***	**0.7279**	0.0166	0.7239	***0.0117***	0.2185	***0.0094***	0.4672
	1:2.25	***0.0088***	0.2509	0.0041	0.2749	***0.0082***	0.2417	0.0700	0.9209	0.0748	0.8830	***0.0101***	**0.6492**	0.0212	0.6816
	1:4	***0.0090***	**0.2682**	0.0048	0.4055	***0.0073***	0.2601	0.1944	0.9522	0.1689	0.9116	0.0412	0.8277	0.0532	0.7432
	1:9	***0.0096***	**0.2826**	0.0168	0.5163	***0.0092***	0.2750	0.3510	0.9754	0.2644	0.9370	0.1348	0.9271	0.0972	0.7956
30:10	1:1	***0.0100***	0.1701	0.0185	0.3087	***0.0091***	0.1627	***0.0141***	0.3386	0.0164	0.3268	***0.0143***	0.3736	***0.0093***	**0.4910**
	1:2.25	0.0220	0.3990	0.0478	0.3807	0.0213	0.3886	0.0430	0.4329	0.0441	0.3910	0.0604	0.4604	0.0798	0.6268
	1:4	0.0532	0.5200	0.0640	0.4032	0.0508	0.5106	0.0739	0.4691	0.0696	0.4131	0.0945	0.4929	0.1464	0.6732
	1:9	0.1270	0.6063	0.0766	0.4024	0.1220	0.6002	0.0986	0.4845	0.0856	0.4123	0.1226	0.5180	0.2165	0.7136
30:30	1:1	0.0047	0.3356	0.0043	0.3308	0.0043	0.3260	***0.0093***	**0.8521**	***0.0103***	0.8308	0.0047	0.6516	***0.0087***	0.8481
	1:2.25	***0.0129***	**0.4983**	***0.0122***	0.4944	***0.0117***	0.4892	0.1223	0.9438	0.1154	0.9159	0.0435	0.8568	0.1184	0.9432
	1:4	0.0223	0.5557	0.0217	0.5477	0.0210	0.5512	0.2258	0.9629	0.1898	0.9382	0.1040	0.9134	0.2208	0.9627
	1:9	0.0424	0.5858	0.0417	0.5740	0.0403	0.5798	0.3577	0.9717	0.2828	0.9439	0.1996	0.9419	0.3550	0.9715
50:100	1:1	***0.0079***	0.6092	***0.0106***	0.6279	***0.0074***	0.6047	***0.0082***	**1.0000**	***0.0085***	0.9999	***0.0078***	0.9883	***0.0074***	0.9994
	1:2.25	0.0042	0.7058	***0.0063***	**0.8062**	0.0042	0.7016	0.3622	1.0000	0.3360	1.0000	0.0589	0.9998	0.2419	1.0000
	1:4	***0.0095***	0.7173	***0.0105***	**0.8439**	***0.0092***	0.7156	0.7481	1.0000	0.6806	1.0000	0.2411	1.0000	0.5775	1.0000
	1:9	***0.0084***	**0.7604**	0.0177	0.8860	***0.0085***	0.7571	0.9124	1.0000	0.8527	1.0000	0.5367	1.0000	0.7951	1.0000
100:50	1:1	***0.0077***	0.6244	***0.0099***	0.6187	***0.0074***	0.6191	***0.0111***	0.9887	***0.0118***	0.9861	***0.0086***	0.9382	***0.0102***	**0.9945**
	1:2.25	0.0215	0.7653	0.0222	0.6819	0.0209	0.7612	0.2116	0.9987	0.1986	0.9970	0.0805	0.9846	0.2911	0.9996
	1:4	0.0381	0.7925	0.0288	0.6861	0.0372	0.7911	0.4207	0.9993	0.3768	0.9985	0.1787	0.9930	0.5500	1.0000
	1:9	0.0778	0.8454	0.0424	0.7311	0.0755	0.8439	0.5922	0.9995	0.5275	0.9980	0.3278	0.9961	0.7315	1.0000
100:100	1:1	***0.0084***	0.7910	***0.0080***	0.7906	***0.0084***	0.7893	***0.0123***	**1.0000**	***0.0131***	**1.0000**	***0.0115***	0.9969	***0.0123***	**1.0000**
	1:2.25	***0.0104***	**0.8765**	***0.0103***	0.8752	**0.0107**	0.8730	0.4558	1.0000	0.4272	1.0000	0.1131	0.9999	0.4530	1.0000
	1:4	0.0202	0.8757	0.0201	0.8741	0.0199	0.8747	0.7719	1.0000	0.7214	1.0000	0.3060	1.0000	0.7691	1.0000
	1:9	0.0241	0.8805	0.0235	0.8781	0.0237	0.8779	0.9060	1.0000	0.8627	1.0000	0.5425	1.0000	0.9052	1.0000

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 2, it could be found that when the populations had log-normal distribution and equal varianceat 0.01 significant level, WBR test and Exact Wilcoxon signed-rank test could control type I errors in all sample sizes. For unequal variance cases t test, t-BSE test and Welch t test could control type I errors in some sample sizes respectively.

It could be found that when the population has log-normal distribution and equal variance, the test statistics with the highest testing power that could control type I errors are WBR test (71.43%) and Exact Wilcoxon signed-rank test(42.86%) respectively. For unequal variance, the test statistics with the highest testing power that could control were t test (28.57%) and Welch t test (9.52%) respectively.

Table 3 The probability of a type I error and power test (0.05 significant level) of statistics on location testing between two populations with gamma distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0440***	0.6170	***0.0381***	0.6109	***0.0484***	0.6269	***0.0531***	**0.7376**	***0.0564***	0.7056	***0.0392***	0.6090	***0.0448***	0.7134
	1:2.25	***0.0546***	0.7129	***0.0499***	0.7003	***0.0582***	0.7199	0.0929	0.8046	0.0912	0.7344	***0.0643***	**0.7417**	0.0804	0.7859
	1:4	***0.0718***	0.7485	***0.0686***	0.7328	***0.0750***	**0.7542**	0.1236	0.8256	0.1066	0.7285	0.0964	0.7765	0.1095	0.8104
	1:9	0.0851	0.7739	0.0800	0.7541	0.0882	0.7789	0.1501	0.8322	0.1157	0.7112	0.1321	0.8062	0.1383	0.8249
10:30	1:1	***0.0437***	0.7771	***0.0632***	0.8277	***0.0456***	0.7812	***0.0530***	**0.9809**	***0.0567***	0.9771	***0.0583***	0.8433	***0.0485***	0.9446
	1:2.25	0.0238	0.8195	***0.0463***	0.9397	0.0248	0.8222	0.1169	0.9964	0.1132	0.9918	***0.0486***	**0.9783**	***0.0645***	0.9737
	1:4	0.0187	0.8489	***0.0411***	**0.9677**	0.0197	0.8513	0.2187	0.9986	0.1933	0.9963	0.0771	0.9944	0.0970	0.9834
	1:9	0.0164	0.8642	***0.0510***	**0.9803**	0.0163	0.8674	0.3386	0.9993	0.2645	0.9980	0.1517	0.9980	0.1367	0.9902
30:10	1:1	***0.0458***	0.7807	***0.0622***	0.7195	***0.0463***	0.7835	***0.0512***	0.7630	***0.0540***	0.7443	***0.0554***	0.7097	***0.0477***	**0.8649**
	1:2.25	0.1044	0.8705	0.0821	0.7378	0.1060	0.8719	0.0906	0.7980	0.0892	0.7617	0.1016	0.7593	0.1477	0.9066
	1:4	0.1580	0.8977	0.0888	0.7484	0.1607	0.8992	0.1133	0.8123	0.1048	0.7626	0.1189	0.7773	0.2007	0.9129
	1:9	0.2215	0.9132	0.0950	0.7554	0.2265	0.9145	0.1498	0.8176	0.1308	0.7347	0.1476	0.7973	0.2677	0.9160
30:30	1:1	***0.0499***	0.9584	***0.0493***	0.9575	***0.0497***	0.9587	***0.0497***	**0.9946**	***0.0518***	0.9927	***0.0463***	0.9750	***0.0497***	**0.9946**
	1:2.25	***0.0522***	0.9781	***0.0514***	0.9774	***0.0530***	**0.9788**	0.1586	0.9982	0.1492	0.9967	0.0786	0.9935	0.1586	0.9982
	1:4	***0.0560***	**0.9838**	***0.0552***	0.9828	***0.0573***	0.9836	0.2526	0.9991	0.2177	0.9980	0.1266	0.9967	0.2527	0.9991
	1:9	***0.0717***	0.9860	***0.0689***	0.9847	***0.0724***	**0.9864**	0.3617	0.9993	0.3006	0.9982	0.2001	0.9985	0.3626	0.9995
50:100	1:1	***0.0484***	0.9999	***0.0486***	**1.0000**	***0.0496***	0.9999	***0.0494***	**1.0000**	***0.0499***	**1.0000**	***0.0457***	**1.0000**	***0.0479***	**1.0000**
	1:2.25	***0.0293***	**1.0000**	***0.0483***	**1.0000**	***0.0292***	**1.0000**	0.3677	1.0000	0.3492	1.0000	0.1052	1.0000	0.2837	1.0000
	1:4	0.0224	1.0000	***0.0494***	**1.0000**	0.0229	1.0000	0.6001	1.0000	0.5497	1.0000	0.2033	1.0000	0.4710	1.0000
	1:9	0.0184	1.0000	***0.0546***	**1.0000**	0.0183	1.0000	0.7853	1.0000	0.7089	1.0000	0.3792	1.0000	0.6446	1.0000
100:50	1:1	***0.0504***	0.9998	***0.0544***	0.9983	***0.0508***	0.9998	***0.0504***	**1.0000**	***0.0505***	**1.0000**	***0.0509***	0.9998	***0.0493***	**1.0000**
	1:2.25	0.0913	1.0000	***0.0594***	**0.9994**	0.0925	1.0000	0.2378	1.0000	0.2259	1.0000	0.1053	1.0000	0.2970	1.0000
	1:4	0.1127	0.9999	***0.0589***	**0.9995**	0.1130	0.9999	0.3967	1.0000	0.3680	1.0000	0.1718	1.0000	0.4877	1.0000
	1:9	0.1424	1.0000	***0.0631***	**0.9996**	0.1435	1.0000	0.4986	1.0000	0.4502	1.0000	0.2632	1.0000	0.6082	1.0000
100:100	1:1	***0.0484***	**1.0000**	***0.0481***	**1.0000**	***0.0490***	**1.0000**	***0.0494***	**1.0000**	***0.0497***	**1.0000**	***0.0478***	**1.0000**	***0.0494***	**1.0000**
	1:2.25	***0.0510***	**1.0000**	***0.0507***	**1.0000**	***0.0511***	**1.0000**	0.3964	1.0000	0.3756	1.0000	0.1207	1.0000	0.3948	1.0000
	1:4	***0.0570***	**1.0000**	***0.0566***	**1.0000**	***0.0582***	**1.0000**	0.6616	1.0000	0.6210	1.0000	0.2539	1.0000	0.6614	1.0000
	1:9	***0.0574***	**1.0000**	***0.0569***	**1.0000**	***0.0578***	**1.0000**	0.8008	1.0000	0.7401	1.0000	0.4189	1.0000	0.8008	1.0000

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 3, it coulb be found that when the populations had gamma distribution and equal variance at 0.05 significant level, all test statistics could control type I errors in all sample sizes. For unequal variance cases Welch t test, t test and t-BSE test could control type I errors in some sample sizes respectively.

It could be found that when the population has gamma distribution and equal variance, the test statistics with the highest testing power that could control type I errors are WBR test (85.71%) and Exact Wilcoxon signed-rank testtest (71.43%) respectively. For unequal variance, the test statistics with the highest testing power that could control type I errors were Welch t test (52.38%), t-BSE test (33.33%) and t test (23.81%) respectively.

Table 4 The probability of a type I error and power test (0.01 significant level) of statistics on location testing between two populations with gamma distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0066***	0.3737	0.0042	0.3553	0.0048	0.3417	***0.0112***	**0.4951**	0.0167	0.4900	***0.0066***	0.3132	***0.0080***	0.4635
	1:2.25	***0.0127***	**0.5221**	***0.0103***	0.5043	***0.0100***	0.4926	0.0248	0.5793	0.0314	0.5334	***0.0132***	0.5071	0.0200	0.5599
	1:4	0.0221	0.5744	0.0202	0.5465	0.0190	0.5473	0.0409	0.6009	0.0428	0.5320	0.0281	0.5716	0.0360	0.5879
	1:9	0.0351	0.6186	0.0324	0.5773	0.0311	0.5961	0.0536	0.6213	0.0494	0.5298	0.0523	0.5996	0.0497	0.6151
10:30	1:1	***0.0086***	0.5563	0.0218	0.5019	***0.0078***	0.5404	***0.0140***	**0.9299**	0.0163	0.9245	0.0167	0.4981	0.0087	0.7703
	1:2.25	***0.0059***	0.6136	***0.0093***	0.8044	***0.0054***	0.6012	0.0370	0.9785	0.0402	0.9658	***0.0086***	0.8715	***0.0124***	**0.8733**
	1:4	***0.0055***	0.6475	***0.0076***	**0.8955**	***0.0052***	0.6332	0.0825	0.9899	0.0775	0.9775	0.0186	0.9578	0.0178	0.8986
	1:9	0.0041	0.6737	***0.0101***	**0.9312**	0.0037	0.6591	0.1612	0.9958	0.1137	0.9838	0.0546	0.9872	0.0270	0.9171
30:10	1:1	***0.0085***	0.5741	0.0181	0.5435	***0.0077***	0.5597	***0.0147***	0.5415	0.0178	0.5204	***0.0147***	0.5185	***0.0096***	**0.6961**
	1:2.25	0.0276	0.7602	0.0329	0.5630	0.0250	0.7542	0.0282	0.5906	0.0290	0.5430	0.0423	0.5570	0.0489	0.7679
	1:4	0.0605	0.8166	0.0394	0.5766	0.0575	0.8102	0.0404	0.6017	0.0375	0.5357	0.0545	0.5688	0.0851	0.7912
	1:9	0.1094	0.8535	0.0445	0.5743	0.1044	0.8492	0.0591	0.6067	0.0520	0.5237	0.0710	0.5824	0.1314	0.8090
30:30	1:1	***0.0082***	0.8731	***0.0073***	0.8716	***0.0078***	0.8669	***0.0106***	**0.9751**	***0.0114***	0.9686	***0.0068***	0.9083	***0.0097***	0.9736
	1:2.25	***0.0125***	**0.9309**	***0.0118***	0.9271	***0.0121***	0.9278	0.0544	0.9893	0.0526	0.9831	0.0222	0.9699	0.0514	0.9889
	1:4	0.0174	0.9462	0.0168	0.9411	0.0167	0.9437	0.1053	0.9935	0.0877	0.9869	0.0472	0.9837	0.1021	0.9933
	1:9	0.0236	0.9572	0.0223	0.9520	0.0230	0.9552	0.1772	0.9943	0.1284	0.9869	0.0887	0.9902	0.1754	0.9942
50:100	1:1	***0.0089***	0.9979	***0.0102***	0.9994	***0.0089***	0.9980	***0.0087***	**1.0000**	***0.0089***	**1.0000**	***0.0084***	**1.0000**	***0.0084***	**1.0000**
	1:2.25	0.0048	0.9999	***0.0091***	**1.0000**	***0.0050***	0.9999	0.1728	1.0000	0.1599	1.0000	0.0262	1.0000	0.1070	1.0000
	1:4	0.0039	1.0000	***0.0118***	**1.0000**	0.0041	0.9999	0.3693	1.0000	0.3149	1.0000	0.0786	1.0000	0.2285	1.0000
	1:9	0.0028	0.9999	***0.0132***	**1.0000**	0.0026	0.9999	0.5880	1.0000	0.4730	1.0000	0.1940	1.0000	0.3760	1.0000
100:50	1:1	***0.0096***	0.9966	***0.0111***	0.9919	***0.0089***	0.9965	***0.0100***	**0.9998**	***0.0108***	**0.9998**	***0.0099***	0.9979	***0.0094***	**0.9998**
	1:2.25	0.0250	0.9992	0.0177	0.9956	0.0259	0.9990	0.0931	1.0000	0.0869	0.9999	0.0363	0.9994	0.1415	1.0000
	1:4	0.0390	0.9997	0.0193	0.9968	0.0383	0.9996	0.1918	1.0000	0.1641	0.9999	0.0742	0.9999	0.2829	1.0000
	1:9	0.0602	0.9997	0.0189	0.9974	0.0588	0.9997	0.2807	0.9999	0.2333	0.9997	0.1267	0.9997	0.4125	1.0000
100:100	1:1	***0.0083***	**1.0000**	***0.0082***	**1.0000**	***0.0084***	**1.0000**	***0.0099***	**1.0000**	***0.0102***	**1.0000**	***0.0081***	**1.0000**	***0.0095***	**1.0000**
	1:2.25	***0.0105***	**1.0000**	***0.0104***	**1.0000**	*0.0105*	1.0000	0.1943	1.0000	0.1787	1.0000	0.0393	1.0000	0.1916	1.0000
	1:4	***0.0140***	**1.0000**	***0.0138***	**1.0000**	*0.0137*	1.0000	0.4350	1.0000	0.3793	1.0000	0.1111	1.0000	0.4316	1.0000
	1:9	0.0157	1.0000	0.0156	1.0000	0.0151	1.0000	0.6054	1.0000	0.5130	1.0000	0.2335	1.0000	0.6033	1.0000

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 4, it could be found that when the populations had gamma distribution and equal variance at 0.01 significant level, The t test, WBR test and Exact Wilcoxon signed-rank test could control type I errors in all sample sizes. For unequal variance cases Welch t test, t-BSE and t test could control type I errors in some sample sizes respectively

It could be found that when the population had gamma distribution and equal variance, the test statistics with the highest testing power that could control type I errors are WBR test(85.71%) and Exact Wilcoxon signed-rank test(57.14%) respectively. For unequal variance, the test statistics with the highest testing power that can control type I errors were Welch t test (33.33%) andt-test (19.05%) respectively.

Table 5 The probability of a type I error and power test (0.05 significant level) of statistics on location testing between two populations with exponential distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0401***	0.6122	***0.0349***	0.6037	***0.0446***	0.6224	***0.0492***	**0.7330**	***0.0525***	0.6972	***0.0362***	0.6034	***0.0403***	0.7054
	1:2.25	***0.0550***	0.7163	***0.0508***	0.7041	***0.0582***	0.7213	0.0927	0.8042	0.0887	0.7353	***0.0642***	**0.7376**	0.0781	0.7836
	1:4	***0.0699***	0.7497	***0.0673***	0.7338	***0.0740***	**0.7558**	0.1215	0.8218	0.1059	0.7323	0.0943	0.7779	0.1076	0.8076
	1:9	0.0895	0.7754	0.0844	0.7545	0.0948	0.7807	0.1550	0.8292	0.1188	0.6989	0.1331	0.8011	0.1438	0.8215
10:30	1:1	***0.0471***	0.7769	***0.0689***	0.8228	***0.0500***	0.7793	***0.0564***	**0.9834**	***0.0588***	0.9786	***0.0613***	0.8476	***0.0523***	0.9464
	1:2.25	***0.0277***	0.8171	***0.0466***	0.9362	***0.0293***	0.8195	0.1263	0.9961	0.1251	0.9926	***0.0488***	**0.9783**	***0.0678***	0.9732
	1:4	0.0187	0.8535	***0.0443***	**0.9664**	0.0195	0.8556	0.2331	0.9985	0.2056	0.9948	0.0801	0.9923	0.1050	0.9829
	1:9	0.0146	0.8712	***0.0510***	**0.9814**	0.0146	0.8750	0.3301	0.9993	0.2594	0.9972	0.1475	0.9980	0.1304	0.9909
30:10	1:1	***0.0446***	0.7867	***0.0648***	0.7263	***0.0462***	0.7895	***0.0570***	0.7734	***0.0606***	0.7553	***0.0629***	0.7137	***0.0531***	**0.8624**
	1:2.25	0.1023	0.8723	0.0837	0.7429	0.1051	0.8738	0.0881	0.8042	0.0855	0.7675	0.0982	0.7639	0.1414	0.9008
	1:4	0.1582	0.9012	0.0895	0.7582	0.1610	0.9018	0.1169	0.8141	0.1076	0.7646	0.1226	0.7841	0.2071	0.9155
	1:9	0.2286	0.9172	0.0932	0.7537	0.2301	0.9191	0.1471	0.8226	0.1235	0.7429	0.1474	0.8012	0.2638	0.9203
30:30	1:1	***0.0487***	0.9504	***0.0478***	0.9497	***0.0496***	0.9520	***0.0524***	**0.9937**	***0.0531***	0.9923	***0.0476***	0.9752	***0.0524***	**0.9937**
	1:2.25	***0.0534***	**0.9771**	***0.0527***	0.9761	**0.0558**	0.9770	0.1669	0.9986	0.1583	0.9975	0.0823	0.9937	0.1669	0.9986
	1:4	***0.0612***	**0.9846**	***0.0601***	0.9830	**0.0625**	0.9838	0.2625	0.9993	0.2296	0.9986	0.1235	0.9976	0.2625	0.9993
	1:9	***0.0693***	**0.9847**	***0.0665***	0.9833	**0.0711**	**0.9847**	0.3629	0.9991	0.3010	0.9975	0.2021	0.9974	0.3640	0.9991
50:100	1:1	***0.0465***	0.9996	***0.0493***	0.9998	***0.0478***	0.9996	***0.0525***	**1.0000**	***0.0529***	**1.0000**	***0.0494***	**1.0000**	***0.0519***	**1.0000**
	1:2.25	***0.0266***	0.9999	***0.0475***	**1.0000**	***0.0272***	0.9999	0.3508	1.0000	0.3300	1.0000	0.0963	1.0000	0.2707	1.0000
	1:4	0.0200	1.0000	***0.0517***	**1.0000**	0.0202	1.0000	0.6078	1.0000	0.5548	1.0000	0.1994	1.0000	0.4739	1.0000
	1:9	0.0157	1.0000	***0.0532***	**1.0000**	0.0158	1.0000	0.7895	1.0000	0.7132	1.0000	0.3822	1.0000	0.6528	1.0000
100:50	1:1	***0.0404***	0.9994	***0.0471***	0.9988	***0.0403***	0.9995	***0.0442***	**1.0000**	***0.0452***	**1.0000**	***0.0463***	**1.0000**	***0.0442***	**1.0000**
	1:2.25	0.0876	1.0000	***0.0594***	**0.9997**	0.0888	1.0000	0.2545	1.0000	0.2437	1.0000	0.1153	1.0000	0.3170	1.0000
	1:4	0.1208	0.9999	***0.0676***	**0.9994**	0.1222	1.0000	0.4120	1.0000	0.3831	1.0000	0.1928	1.0000	0.5051	1.0000
	1:9	0.1472	1.0000	***0.0651***	**0.9995**	0.1477	1.0000	0.4924	1.0000	0.4451	1.0000	0.2534	1.0000	0.6060	1.0000
100:100	1:1	***0.0496***	**1.0000**	***0.0494***	**1.0000**	***0.0502***	**1.0000**	***0.0474***	**1.0000**	***0.0479***	**1.0000**	***0.0470***	**1.0000**	***0.0473***	**1.0000**
	1:2.25	***0.0516***	**1.0000**	***0.0514***	**1.0000**	***0.0538***	**1.0000**	0.4266	1.0000	0.4071	1.0000	0.1439	1.0000	0.4259	1.0000
	1:4	***0.0533***	**1.0000**	***0.0528***	**1.0000**	***0.0540***	**1.0000**	0.6389	1.0000	0.5952	1.0000	0.2398	1.0000	0.6389	1.0000
	1:9	***0.0578***	**1.0000**	***0.0571***	**1.0000**	***0.0583***	**1.0000**	0.7698	1.0000	0.7096	1.0000	0.3734	1.0000	0.7699	1.0000

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 5, it could be found that when the populations have exponential distribution and equal variance at 0.05 significant level, all test statistics can control type I errors in all sample sizes. For unequal variance cases Welch t testcan control type I errors in almost all sample sizes.

It could be found that when the population has exponential distribution and equal variance, the test statistics with the highest testing power that can control type I errors are WBR test (85.71%), and Exact Wilcoxon signed-rank test(71.43%)respectively. For unequal variance, the test statistics with the highest testing power that could control were Welch t test (52.38%).

Table 6 The probability of a type I error and power test (0.01 significant level) of statistics on location testing between two populations which have exponential distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0057***	0.3708	0.0035	0.3524	0.0037	0.3365	***0.0096***	0.4905	***0.0142***	**0.4906**	***0.0060***	0.3102	***0.0071***	0.4606
	1:2.25	***0.0122***	**0.5215**	***0.0094***	0.5028	***0.0100***	0.4931	0.0248	0.5769	0.0318	0.5335	***0.0139***	0.5077	0.0197	0.5561
	1:4	0.0218	0.5793	0.0193	0.5516	0.0181	0.5557	0.0409	0.6039	0.0431	0.5281	0.0293	0.5688	0.0362	0.5941
	1:9	0.0333	0.6243	0.0302	0.5791	0.0289	0.6003	0.0530	0.6164	0.0471	0.5221	0.0530	0.6052	0.0483	0.6087
10:30	1:1	***0.0094***	0.5627	***0.0218***	0.4975	***0.0088***	0.5463	***0.0150***	**0.9300**	0.0180	0.9252	0.0164	0.4979	***0.0102***	0.7682
	1:2.25	***0.0063***	0.6129	***0.0078***	0.7979	***0.0059***	0.5987	0.0339	0.9772	0.0407	0.9645	***0.0107***	0.8690	***0.0110***	**0.8636**
	1:4	0.0040	0.6592	***0.0071***	**0.8943**	0.0039	0.6450	0.0861	0.9891	0.0799	0.9771	0.0174	0.9587	0.0175	0.9004
	1:9	0.0023	0.6878	***0.0105***	**0.9372**	0.0019	0.6749	0.1524	0.9952	0.1072	0.9854	0.0499	0.9877	0.0278	0.9229
30:10	1:1	***0.0089***	0.5766	0.0196	0.5510	***0.0082***	0.5630	***0.0143***	0.5459	0.0174	0.5292	0.0170	0.5190	***0.0093***	**0.6996**
	1:2.25	0.0305	0.7657	0.0343	0.5729	0.0280	0.7577	0.0297	0.5981	0.0307	0.5480	0.0414	0.5694	0.0487	0.7737
	1:4	0.0587	0.8254	0.0410	0.5842	0.0544	0.8203	0.0423	0.6123	0.0391	0.5490	0.0552	0.5854	0.0870	0.7959
	1:9	0.1131	0.8563	0.0476	0.5844	0.1088	0.8502	0.0580	0.6095	0.0505	0.5214	0.0697	0.5921	0.1288	0.8128
30:30	1:1	***0.0093***	0.8677	***0.0085***	0.8655	***0.0087***	0.8634	***0.0107***	0.9742	***0.0124***	0.9668	***0.0073***	0.9117	***0.0098***	**0.9730**
	1:2.25	***0.0122***	0.9246	***0.0118***	0.9208	***0.0117***	0.9229	0.0559	0.9905	0.0554	0.9822	0.0247	0.9674	0.0534	0.9902
	1:4	0.0199	0.9439	0.0196	0.9390	0.0196	0.9405	0.1096	0.9941	0.0899	0.9878	0.0459	0.9838	0.1059	0.9940
	1:9	0.0234	0.9503	0.0229	0.9452	0.0227	0.9487	0.1806	0.9938	0.1343	0.9840	0.0965	0.9876	0.1785	0.9937
50:100	1:1	***0.0112***	0.9969	***0.0108***	0.9992	***0.0106***	0.9971	***0.0109***	**1.0000**	***0.0116***	**1.0000**	***0.0101***	**1.0000**	***0.0106***	**1.0000**
	1:2.25	***0.0052***	0.9991	***0.0092***	**0.9999**	0.0049	0.9991	0.1624	1.0000	0.1522	1.0000	0.0254	1.0000	0.1010	1.0000
	1:4	0.0037	0.9999	***0.0100***	**1.0000**	0.0036	0.9999	0.3749	1.0000	0.3166	1.0000	0.0736	1.0000	0.2239	1.0000
	1:9	0.0030	1.0000	***0.0117***	**1.0000**	0.0030	1.0000	0.5970	1.0000	0.4804	1.0000	0.2006	1.0000	0.3879	1.0000
100:50	1:1	***0.0076***	0.9968	***0.0099***	0.9911	***0.0077***	0.9964	***0.0077***	0.9999	***0.0084***	0.9999	***0.0081***	0.9965	***0.0073***	**1.0000**
	1:2.25	0.0231	0.9997	0.0169	0.9964	0.0231	0.9996	0.1064	1.0000	0.0988	1.0000	0.0405	0.9997	0.1536	1.0000
	1:4	0.0428	0.9995	0.0199	0.9954	0.0417	0.9995	0.2139	0.9999	0.1858	0.9999	0.0821	0.9998	0.3082	1.0000
	1:9	0.0615	0.9997	0.0198	0.9967	0.0605	0.9997	0.2767	0.9999	0.2263	0.9999	0.1212	0.9999	0.4098	1.0000
100:100	1:1	**0.0101**	**1.0000**	**0.0100**	**1.0000**	**0.0102**	**1.0000**	**0.0098**	**1.0000**	**0.0100**	**1.0000**	**0.0074**	**1.0000**	**0.0093**	**1.0000**
	1:2.25	**0.0118**	**1.0000**	**0.0118**	**1.0000**	**0.0114**	**1.0000**	0.2214	1.0000	0.2068	1.0000	0.0524	1.0000	0.2190	1.0000
	1:4	**0.0140**	**1.0000**	**0.0135**	**1.0000**	**0.0132**	**1.0000**	0.4085	1.0000	0.3548	1.0000	0.1084	1.0000	0.4053	1.0000
	1:9	0.0154	1.0000	0.0153	1.0000	**0.0150**	**1.0000**	0.5691	1.0000	0.4708	1.0000	0.1912	1.0000	0.5671	1.0000

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 6, it could be found that when the populations had exponential distribution and equal varianceat 0.01 significant level, t test,WBR test and Exact Wilcoxon signed-rank test could control type I errors in all sample sizes. For unequal variance cases Welch t test,t test and t-BSE testcan control type I errors in some sample sizes respectively.

It could be found that when a population had exponential distribution and equal variance, the test statistics with the highest testing power that could control type I errors are Exact Wilcoxon signed-rank test (71.43%), WBR test(42.86%) and B-M test (42.86%) respectively. For unequal variance, the test statistics with the highest testing power that could control were Welch t test(23.81%) and t test (19.05%)respectively.

Table 7 The probability of a type I error and power test (0.05 significant level) of statistics on location testing between two populations with poisson distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0493***	0.1558	***0.0478***	0.1522	***0.0533***	0.1637	***0.0511***	0.1588	***0.0569***	**0.1728**	***0.0469***	0.1286	***0.0447***	0.1433
	1:2.25	***0.0516***	0.2108	***0.0489***	0.2052	***0.0553***	0.2215	***0.0558***	**0.2258**	***0.0571***	0.2234	***0.0523***	0.1772	***0.0493***	0.2034
	1:4	***0.0567***	0.2586	***0.0522***	0.2488	***0.0591***	0.2659	***0.0635***	**0.2744**	***0.0590***	0.2545	***0.0549***	0.2209	***0.0554***	0.2553
	1:9	***0.0602***	0.2865	***0.0526***	0.2652	***0.0641***	0.2943	***0.0716***	**0.3013**	***0.0562***	0.2535	***0.0608***	0.2362	***0.0648***	0.2865
10:30	1:1	***0.0481***	0.2130	***0.0525***	0.1816	***0.0504***	0.2169	***0.0526***	0.2127	***0.0552***	**0.2221**	***0.0578***	0.1682	***0.0447***	0.1938
	1:2.25	0.0181	0.2252	***0.0504***	0.3327	0.0186	0.2300	***0.0609***	0.3895	***0.0599***	**0.3901**	***0.0581***	0.3075	***0.0324***	0.2844
	1:4	0.0107	0.2088	***0.0485***	0.4197	0.0113	0.2130	***0.0683***	**0.5134**	***0.0603***	0.4916	***0.0528***	0.3973	0.0224	0.3220
	1:9	0.0047	0.2034	***0.0528***	0.5461	0.0050	0.2082	0.0911	0.6484	***0.0636***	**0.5760**	***0.0578***	0.5333	0.0207	0.3775
30:10	1:1	***0.0482***	0.2283	***0.0520***	**0.2364**	***0.0496***	0.2324	***0.0534***	0.2129	***0.0562***	0.2229	***0.0585***	0.2060	***0.0464***	0.2143
	1:2.25	0.1050	0.4025	***0.0525***	0.2671	0.1072	0.4067	***0.0567***	**0.2698**	***0.0561***	0.2657	***0.0592***	0.2336	0.0779	0.3464
	1:4	0.1522	0.4728	***0.0548***	0.2672	0.1538	0.4763	***0.0597***	**0.2824**	***0.0570***	0.2679	***0.0627***	0.2285	0.0994	0.3848
	1:9	0.2154	0.5533	***0.0532***	0.2707	0.2206	0.5573	***0.0666***	**0.2964**	***0.0597***	0.2684	***0.0605***	0.2374	0.1357	0.4360
30:30	1:1	***0.0493***	0.4037	***0.0493***	0.4032	***0.0511***	**0.4073**	***0.0493***	0.3934	***0.0510***	0.4022	***0.0504***	0.3403	***0.0476***	0.3875
	1:2.25	***0.0503***	0.5056	***0.0497***	0.5023	***0.0517***	0.5104	***0.0562***	**0.5505**	***0.0531***	0.5383	***0.0525***	0.4656	***0.0543***	0.5461
	1:4	***0.0499***	0.5930	***0.0485***	0.5865	***0.0501***	0.5965	***0.0633***	**0.6464**	***0.0555***	0.6150	***0.0521***	0.5505	***0.0620***	0.6425
	1:9	***0.0512***	0.6373	***0.0491***	0.6271	***0.0534***	0.6387	0.0807	0.6964	***0.0614***	0.6393	***0.0560***	0.6097	0.0797	0.6946
50:100	1:1	***0.0539***	0.7265	***0.0531***	0.7269	***0.0548***	0.7275	***0.0618***	0.7294	***0.0593***	**0.7328**	***0.0574***	0.6520	***0.0606***	0.7213
	1:2.25	***0.0268***	0.8294	***0.0490***	0.8951	**0.0272**	0.8301	***0.0601***	**0.9300**	***0.0558***	0.9250	***0.0545***	0.8639	***0.0439***	0.9009
	1:4	0.0180	0.8706	***0.0537***	0.9466	0.0177	0.8719	0.0834	0.9720	***0.0703***	**0.9646**	***0.0635***	0.9362	***0.0481***	0.9490
	1:9	0.0108	0.9215	***0.0471***	**0.9794**	0.0110	0.9213	0.1224	0.9907	0.0869	0.9836	0.0760	0.9794	***0.0617***	0.9766
100:50	1:1	0.1052	0.7200	***0.0537***	**0.7097**	0.1072	0.7197	***0.0568***	0.6991	***0.0565***	0.7018	***0.0630***	0.6318	0.0795	0.7091
	1:2.25	0.0860	0.8558	***0.0519***	0.7880	0.0872	0.8571	***0.0625***	**0.8236**	***0.0593***	0.8157	***0.0555***	0.7441	0.0792	0.8625
	1:4	0.1158	0.8963	***0.0531***	0.8059	0.1153	0.8963	***0.0677***	**0.8503**	***0.0605***	0.8327	***0.0582***	0.7831	0.1042	0.8980
	1:9	0.1384	0.9338	***0.0521***	0.8430	0.1379	0.9338	0.0870	0.8748	***0.0719***	0.8471	***0.0666***	0.8377	0.1417	0.9230
100:100	1:1	***0.0287***	**0.8730**	***0.0530***	**0.8730**	***0.0278***	0.8727	***0.0619***	0.8647	***0.0577***	0.8662	***0.0564***	0.7993	***0.0440***	0.8644
	1:2.25	***0.0504***	0.9498	***0.0503***	0.9495	***0.0505***	0.9505	***0.0667***	**0.9703**	***0.0623***	0.9665	***0.0593***	0.9336	***0.0665***	0.9702
	1:4	***0.0527***	0.9795	***0.0523***	0.9792	***0.0534***	**0.9796**	0.0992	0.9898	0.0879	0.9856	*0.0739*	0.9728	0.0987	0.9898
	1:9	***0.0502***	**0.9829**	***0.0494***	0.9826	***0.0509***	0.9828	0.1112	0.9923	0.0815	0.9878	0.0754	0.9851	0.1111	0.9923

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 7, it could be found that when the populations have poisson distribution and equal variance at 0.05 significant level, Welch t test, WBR test, B-M test and Y-W testcan control type I errors in all sample sizes. For unequal variance cases Welch t test, could control type I errors in all sample sizes followed byY-W test,B-Mtest,WBR testrespectively.

It could be found that when a population had poisson distribution and equal variance, the test statistics with the highest testing power that could control type I errors were Welch t test (42.86%) and WBR test (42.86%). For unequal variance, the test statistics with the highest testing power that could control almost all sample sizes was WBR test (61.90%).

Table 8 The probability of a type I error and power test (0.01 significant level) of statistics on location testing between two populations with poisson distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0093***	0.0500	***0.0089***	0.0468	0.0076	0.0412	***0.0103***	**0.0535**	0.0173	0.0733	***0.0102***	0.0325	***0.0073***	0.0349
	1:2.25	***0.0107***	0.0750	***0.0101***	0.0708	***0.0092***	0.0656	***0.0146***	**0.0837**	0.0185	0.0986	***0.0101***	0.0569	***0.0086***	0.0598
	1:4	***0.0134***	0.1047	***0.0115***	0.0950	***0.0109***	0.0887	***0.0150***	**0.1152**	0.0195	0.1228	***0.0135***	0.0802	***0.0092***	0.0830
	1:9	0.0164	0.1322	***0.0124***	0.1114	***0.0136***	**0.1196**	0.0207	0.1421	0.0190	0.1304	0.0184	0.1010	***0.0135***	0.1127
10:30	1:1	***0.0092***	0.0768	***0.0107***	0.0568	***0.0081***	0.0719	***0.0125***	**0.0856**	0.0163	0.0990	0.0155	0.0570	***0.0079***	0.0592
	1:2.25	***0.0014***	0.0732	***0.0092***	0.1380	0.0011	0.0680	***0.0139***	0.1906	***0.0142***	**0.2023**	***0.0135***	0.1214	0.0028	0.0931
	1:4	0.0007	0.0602	***0.0097***	**0.1973**	0.0006	0.0547	0.0154	0.2774	0.0155	0.2716	***0.0126***	0.1732	0.0028	0.1026
	1:9	0.0001	0.0515	***0.0106***	**0.3013**	0.0001	0.0484	0.0242	0.4127	0.0158	0.3410	*0.0139*	0.2779	0.0024	0.1272
30:10	1:1	***0.0086***	0.0752	***0.0106***	**0.0916**	***0.0077***	0.0713	***0.0122***	0.0809	0.0156	0.0962	***0.0146***	0.0787	***0.0072***	0.0671
	1:2.25	0.0312	0.2209	***0.0117***	0.1123	0.0292	0.2118	***0.0139***	**0.1227**	0.0163	0.1259	0.0175	0.1032	0.0178	0.1610
	1:4	0.0599	0.3036	***0.0141***	**0.1087**	0.0573	0.2938	0.0194	0.1288	0.0204	0.1261	0.0219	0.1048	0.0301	0.1996
	1:9	0.1012	0.3971	***0.0134***	**0.1153**	0.0974	0.3874	0.0224	0.1451	0.0200	0.1313	0.0225	0.1072	0.0462	0.2500
30:30	1:1	***0.0097***	0.1809	***0.0094***	0.1801	***0.0094***	0.1748	***0.0110***	0.1821	***0.0128***	**0.1993**	***0.0117***	0.1429	***0.0097***	0.1675
	1:2.25	***0.0114***	0.2673	***0.0113***	0.2630	***0.0103***	0.2611	***0.0137***	**0.3103**	***0.0138***	0.3076	***0.0111***	0.2332	***0.0121***	0.2929
	1:4	***0.0118***	0.3518	***0.0110***	0.3423	***0.0111***	0.3450	0.0164	0.4073	***0.0147***	0.3795	***0.0144***	0.3104	***0.0147***	**0.3922**
	1:9	***0.0135***	**0.4031**	***0.0118***	0.3833	***0.0125***	0.3940	0.0223	0.4707	0.0154	0.4003	***0.0150***	0.3725	0.0206	0.4589
50:100	1:1	***0.0111***	0.4913	***0.0108***	0.4836	***0.0105***	0.4865	0.0152	0.4986	***0.0150***	**0.5113**	***0.0136***	0.3927	***0.0140***	0.4782
	1:2.25	0.0034	0.6090	***0.0093***	0.7375	0.0035	0.6046	***0.0138***	**0.8084**	***0.0129***	0.7998	***0.0130***	0.6755	***0.0072***	0.7336
	1:4	0.0014	0.6573	***0.0097***	**0.8345**	0.0014	0.6521	0.0227	0.9066	0.0183	0.8860	0.0154	0.8013	***0.0074***	0.8213
	1:9	0.0002	0.7368	***0.0100***	**0.9211**	0.0002	0.7285	0.0390	0.9646	0.0230	0.9383	0.0190	0.9159	***0.0118***	0.9049
100:50	1:1	0.0339	0.4928	***0.0115***	**0.4877**	0.0316	0.4886	***0.0147***	0.4657	0.0166	0.4761	***0.0191***	0.3962	0.0184	0.4741
	1:2.25	0.0244	0.6997	***0.0122***	0.5757	0.0242	0.6976	0.0153	0.6217	***0.0141***	**0.6089**	***0.0131***	0.4996	0.0227	0.6937
	1:4	0.0392	0.7763	***0.0107***	**0.6021**	0.0377	0.7733	0.0183	0.6647	0.0156	0.6359	***0.0132***	0.5485	0.0323	0.7607
	1:9	0.0539	0.8517	***0.0111***	**0.6442**	0.0524	0.8482	0.0237	0.7139	0.0178	0.6630	***0.0148***	0.6126	0.0528	0.8189
100:100	1:1	0.0027	0.6930	***0.0111***	0.6930	0.0027	0.6888	***0.0131***	0.6852	***0.0124***	**0.6938**	***0.0131***	0.5751	***0.0075***	0.6801
	1:2.25	***0.0094***	0.8517	***0.0094***	0.8507	***0.0091***	0.8492	0.0153	0.8954	***0.0136***	**0.8873**	***0.0134***	0.8047	0.0151	0.8931
	1:4	***0.0111***	**0.9200**	***0.0109***	0.9185	***0.0105***	0.9180	0.0287	0.9559	0.0230	0.9441	0.0205	0.8984	0.0280	0.9547
	1:9	***0.0109***	**0.9388**	***0.0105***	0.9371	***0.0106***	0.9379	0.0343	0.9677	0.0202	0.9515	0.0194	0.9358	0.0337	0.9672

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 8, it coulde be found that when the populations had poisson distribution and equal varianceat 0.01 significant level, Welch t test could control type I errors in all sample sizes followed by WBR test,Y-W testandExact Wilcoxon signed-rank test. For unequal variance cases Welch t test could control type I errors in all sample sizes.

It could be found that when the population has poisson distribution and equal variance, the test statistics with the highest testing power that could control type I errors was B-M test (42.86%). For unequal variance, the test statistics with the highest testing power that could control type I errors was Welch ttest (38.10%).

Table **9** The probability of a type I error and power test (0.05 significant level) of statistics on location testing between two populations with uniform distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0506***	0.4267	***0.0494***	0.4241	***0.0544***	**0.4379**	***0.0521***	0.4009	***0.0547***	0.4246	***0.0541***	0.2523	***0.0429***	0.3668
	1:2.25	***0.0546***	0.5620	***0.0529***	0.5516	***0.0573***	**0.5740**	***0.0590***	0.5014	***0.0570***	0.5105	***0.0592***	0.3355	***0.0496***	0.4674
	1:4	***0.0595***	0.6312	***0.0548***	0.6072	***0.0631***	**0.6416**	***0.0678***	0.5213	***0.0564***	0.5050	***0.0676***	0.3724	***0.0572***	0.4882
	1:9	***0.0624***	0.6804	***0.0547***	0.6386	***0.0655***	**0.6899**	***0.0746***	0.4949	***0.0524***	0.4331	0.0769	0.3798	***0.0690***	0.4743
10:30	1:1	***0.0490***	0.6206	***0.0531***	0.5763	***0.0496***	**0.6248**	***0.0534***	0.5434	***0.0560***	0.5634	***0.0684***	0.3527	***0.0496***	0.5541
	1:2.25	0.0190	0.7385	***0.0513***	**0.8463**	0.0204	0.7419	***0.0560***	0.7634	***0.0501***	0.7727	***0.0587***	0.5847	***0.0257***	0.6919
	1:4	0.0088	0.7871	***0.0477***	**0.9406**	0.0089	0.7908	***0.0634***	0.8428	***0.0485***	0.8384	***0.0530***	0.7309	0.0179	0.7247
	1:9	0.0043	0.8401	***0.0471***	**0.9846**	0.0045	0.8435	0.0797	0.9003	***0.0495***	0.8643	***0.0515***	0.8543	0.0158	0.7303
30:10	1:1	***0.0500***	0.6320	***0.0511***	0.5849	***0.0512***	**0.6364**	***0.0518***	0.5487	***0.0537***	0.5699	***0.0653***	0.3582	***0.0501***	0.5628
	1:2.25	0.1083	0.8226	***0.0538***	**0.6411**	0.1099	0.8250	***0.0566***	0.5908	***0.0569***	0.5942	***0.0751***	0.3900	0.0934	0.6682
	1:4	0.1495	0.8789	***0.0507***	**0.6531**	0.1524	0.8800	***0.0571***	0.5552	***0.0521***	0.5428	***0.0700***	0.3929	0.1140	0.6779
	1:9	0.2110	0.9231	***0.0592***	**0.6612**	0.2138	0.9242	***0.0671***	0.5064	***0.0545***	0.4671	0.0759	0.3912	0.1431	0.6641
30:30	1:1	***0.0500***	0.9132	***0.0498***	0.9132	***0.0498***	**0.9146**	***0.0465***	0.8653	***0.0496***	0.8728	***0.0526***	0.6887	***0.0465***	0.8653
	1:2.25	***0.0505***	0.9749	***0.0496***	0.9746	***0.0519***	**0.9760**	***0.0530***	0.9320	***0.0492***	0.9327	***0.0522***	0.8230	***0.0530***	0.9320
	1:4	***0.0490***	0.9910	***0.0471***	0.9901	***0.0513***	**0.9911**	***0.0578***	0.9408	***0.0465***	0.9343	***0.0493***	0.8782	***0.0578***	0.9408
	1:9	***0.0495***	**0.9938**	***0.0464***	0.9933	***0.0499***	0.9935	***0.0693***	0.9243	***0.0449***	0.9016	***0.0503***	0.9038	***0.0696***	0.9245
50:100	1:1	***0.0500***	0.9992	***0.0505***	0.9991	***0.0500***	**0.9993**	***0.0505***	0.9958	***0.0511***	0.9959	***0.0537***	0.9580	***0.0495***	0.9958
	1:2.25	***0.0260***	**1.0000**	***0.0493***	**1.0000**	***0.0264***	**1.0000**	***0.0554***	0.9999	***0.0493***	0.9999	***0.0529***	0.9970	***0.0322***	0.9998
	1:4	0.0168	1.0000	***0.0518***	**1.0000**	0.0164	1.0000	***0.0725***	0.9999	***0.0528***	0.9999	***0.0544***	0.9995	***0.0318***	0.9997
	1:9	0.0107	1.0000	***0.0500***	**1.0000**	0.0108	1.0000	0.0838	1.0000	***0.0483***	**1.0000**	***0.0517***	**1.0000**	***0.0327***	**1.0000**
100:50	1:1	***0.0488***	0.9989	***0.0486***	**0.9990**	***0.0492***	0.9989	***0.0502***	0.9967	***0.0505***	0.9968	***0.0504***	0.9633	***0.0479***	0.9963
	1:2.25	0.0870	1.0000	***0.0478***	**0.9999**	0.0862	1.0000	***0.0530***	0.9980	***0.0486***	0.9978	***0.0509***	0.9849	0.0784	0.9986
	1:4	0.1174	1.0000	***0.0527***	**0.9999**	0.1180	1.0000	***0.0618***	0.9963	***0.0524***	0.9960	***0.0536***	0.9888	0.1004	0.9978
	1:9	0.1380	1.0000	***0.0528***	**1.0000**	0.1397	1.0000	***0.0669***	0.9944	***0.0515***	0.9919	***0.0542***	0.9940	0.1225	0.9967
100:100	1:1	***0.0504***	**1.0000**	***0.0504***	**1.0000**	***0.0507***	**1.0000**	***0.0496***	0.9997	***0.0499***	0.9997	***0.0494***	0.9951	***0.0491***	0.9997
	1:2.25	***0.0481***	**1.0000**	***0.0479***	**1.0000**	***0.0488***	**1.0000**	***0.0548***	**1.0000**	***0.0484***	**1.0000**	***0.0496***	0.9999	***0.0545***	**1.0000**
	1:4	***0.0519***	**1.0000**	***0.0516***	**1.0000**	***0.0542***	**1.0000**	***0.0648***	**1.0000**	***0.0512***	**1.0000**	***0.0532***	**1.0000**	***0.0647***	**1.0000**
	1:9	***0.0495***	**1.0000**	***0.0487***	**1.0000**	***0.0504***	**1.0000**	***0.0707***	**1.0000**	***0.0475***	0.9999	***0.0488***	0.9999	***0.0707***	**1.0000**

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 9, it could be found that when the populations have uniform distributionand equal variance at 0.05 significant level, all test statistics could control type I errors in all sample sizes. For unequal variance cases, Welch t test and B-Mtest could control type I errors in all sample sizes followed by WBR test and Y-Wtest respectively.

It could be found that when a population had uniform distributionand equal variance, the test statistics with the highest testing power that could control type I errors were t-BSE test (85.71%). For unequal variance, the test statistics with the highest testing power that could control type I errors was Welch t test (71.43%).

Table 10 The probability of a type I error and power test (0.01 significant level) of statistics on location testing between two populations with uniform distribution classified by sample size and variance

Sample Size	Var.ratio	t		Welch		t-BSE		WBR		B-M		Y-W		Wilcoxon	
		TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT	TE	PT
10:10	1:1	***0.0124***	**0.1853**	***0.0116***	0.1801	***0.0099***	0.1595	***0.0118***	0.1769	0.0170	0.2430	***0.0148***	0.0838	***0.0093***	0.1511
	1:2.25	***0.0143***	**0.2890**	***0.0130***	0.2712	***0.0117***	0.2561	***0.0140***	0.2554	0.0176	0.3128	0.0152	0.1323	***0.0118***	0.2260
	1:4	0.0160	0.3432	***0.0138***	0.3028	***0.0135***	**0.3059**	0.0185	0.2752	0.0199	0.3001	0.0192	0.1523	***0.0150***	0.2463
	1:9	0.0205	0.3858	0.0166	0.3147	0.0170	0.3478	0.0220	0.2727	0.0200	0.2565	0.0236	0.1654	0.0198	0.2550
10:30	1:1	***0.0095***	**0.3529**	***0.0135***	0.2900	***0.0085***	0.3390	***0.0118***	0.2931	***0.0150***	0.3324	0.0241	0.1574	***0.0088***	0.2922
	1:2.25	0.0032	0.4184	***0.0140***	**0.6042**	0.0023	0.3998	0.0158	0.5159	0.0152	0.5650	0.0165	0.3090	0.0037	0.3871
	1:4	0.0010	0.4500	***0.0099***	**0.7917**	0.0010	0.4285	0.0156	0.6363	***0.0126***	0.6553	***0.0123***	0.4456	0.0026	0.4087
	1:9	0.0002	0.4728	***0.0109***	**0.9228**	0.0001	0.4514	0.0228	0.7367	***0.0110***	0.6882	***0.0131***	0.6138	0.0019	0.4076
30:10	1:1	***0.0096***	**0.3659**	***0.0130***	0.2935	***0.0094***	0.3473	***0.0132***	0.2939	0.0158	0.3363	0.0180	0.1591	***0.0087***	0.2953
	1:2.25	0.0362	0.6335	***0.0146***	**0.3224**	0.0339	0.6205	0.0156	0.3368	0.0181	0.3537	0.0265	0.1756	0.0252	0.4226
	1:4	0.0622	0.7469	***0.0134***	**0.3295**	0.0587	0.7329	0.0156	0.3262	0.0153	0.3222	0.0218	0.1695	0.0345	0.4523
	1:9	0.1086	0.8266	0.0171	0.3240	0.1059	0.8171	0.0230	0.2919	0.0204	0.2688	0.0257	0.1621	0.0563	0.4618
30:30	1:1	***0.0096***	**0.7514**	***0.0096***	0.7512	***0.0091***	0.7413	***0.0098***	0.6779	***0.0117***	0.7090	***0.0101***	0.4263	***0.0088***	0.6665
	1:2.25	***0.0106***	**0.9009**	***0.0106***	0.8979	***0.0100***	0.8968	***0.0115***	0.8042	***0.0103***	0.8168	***0.0120***	0.5951	***0.0106***	0.7971
	1:4	***0.0096***	**0.9488**	***0.0091***	0.9443	***0.0090***	0.9454	***0.0135***	0.8257	***0.0104***	0.8139	***0.0110***	0.6699	***0.0130***	0.8191
	1:9	***0.0104***	**0.9681**	***0.0087***	0.9629	***0.0096***	0.9651	0.0163	0.8028	***0.0091***	0.7552	***0.0127***	0.7045	0.0159	0.8000
50:100	1:1	***0.0109***	**0.9912**	***0.0116***	0.9909	***0.0113***	**0.9912**	***0.0110***	0.9762	***0.0117***	0.9785	***0.0111***	0.8622	***0.0110***	0.9742
	1:2.25	0.0032	0.9997	***0.0092***	**1.0000**	0.0034	0.9996	***0.0122***	0.9986	***0.0104***	0.9986	***0.0104***	0.9813	0.0048	0.9970
	1:4	0.0016	0.9998	***0.0095***	**1.0000**	0.0018	0.9998	0.0154	0.9995	***0.0094***	0.9991	***0.0107***	0.9958	0.0036	0.9981
	1:9	0.0007	1.0000	***0.0103***	**1.0000**	0.0007	1.0000	0.0236	0.9999	***0.0108***	0.9995	***0.0116***	0.9996	***0.0051***	0.9984
100:50	1:1	***0.0095***	0.9930	***0.0098***	**0.9932**	***0.0092***	0.9928	***0.0100***	0.9808	***0.0111***	0.9823	***0.0117***	0.8624	***0.0092***	0.9788
	1:2.25	0.0231	0.9993	***0.0098***	**0.9982**	0.0221	0.9991	***0.0116***	0.9899	***0.0102***	0.9902	***0.0107***	0.9249	0.0196	0.9932
	1:4	0.0383	0.9999	***0.0099***	**0.9992**	0.0375	0.9998	***0.0148***	0.9809	***0.0113***	0.9786	***0.0141***	0.9434	0.0309	0.9895
	1:9	0.0551	1.0000	***0.0100***	**1.0000**	0.0536	1.0000	0.0171	0.9705	***0.0110***	0.9614	***0.0118***	0.9609	0.0419	0.9886
100:100	1:1	***0.0095***	**0.9996**	***0.0095***	**0.9996**	***0.0088***	**0.9996**	***0.0095***	0.9979	***0.0100***	0.9980	***0.0095***	0.9726	***0.0093***	0.9978
	1:2.25	***0.0101***	**1.0000**	***0.0099***	**1.0000**	***0.0093***	**1.0000**	***0.0118***	**1.0000**	***0.0106***	**1.0000**	***0.0109***	0.9977	***0.0115***	**1.0000**
	1:4	***0.0099***	**1.0000**	***0.0098***	**1.0000**	***0.0094***	**1.0000**	***0.0142***	**1.0000**	***0.0101***	**1.0000**	***0.0105***	0.9995	***0.0139***	**1.0000**
	1:9	***0.0098***	**1.0000**	***0.0094***	**1.0000**	***0.0099***	**1.0000**	0.0183	0.9994	***0.0102***	0.9994	***0.0112***	0.9995	0.0181	0.9994

* Italic and bold indicate the case that was able to control type I error.

** Bold entries indicate the highest testing power controlled under a type I error.

From table 10, it could be found that when the populations have uniform distribution and equal varianceat 0.01 significant level, all test statistics can control type I errors in all sample sizes except B-Mtest and Y-W test. For unequal variance cases Welch t testcan control type I errors in almost sample sizes followed by BM test and YWtest.

It could be found that when the population has uniform distribution and equal variance, the test statistics with the highest testing power that could control type I errors is t test (85.71%). For unequal variance, the test statistics with the highest testing power that could control some sample sizes was Welch t test (66.67%).

Discussion

The results revealed that when the populations had five distributions on equal variance, all test statistics could control the type I error on almost sample sizes. This means that the said test statistics were robust. Nonetheless, when the variance was unequal, it was found that all test statistics had a lower capacity in controlling the type I error. Specifically, when the populations had log-normal distribution, gamma distributionand exponential distribution on unequal variance ratio was 1:9, it could control the type I error on some sample sizes. Moreover, the results also showed that the test statistics of WBR test, B-M test, Y-W test, and Exact Wilcoxon signed-rank test could not control the type I error when the variance was unequal in almost all conditions. The above test statistics were proposed to solve the problem of Behrences-Fisher. As shown in table 4, the ratio of variances was 1:9 with the sample size of 100,100 and the type I error was found at above 23.35%. Comparing with Welch t test, the type I error was found at approximately 1.56%. In particular, the Exact Wilcoxon signed-rank teststatistic is not suitable for hypotheses testing when the variance is unequal (Harwell et al., 1992; Zimmerman & Zumbo, 1993a; Zimmerman & Zumbo, 1993b; Stonehouse & Forrester, 1998). Furthermore, the test statistics with the highest testing power that could control the type I error for equal variance was WBR while for unequal variance was Welch t test. This maybe caused by the said location testing was the test of the difference of the mean which, the Welch t test t was therefore more efficient than other test statistics.

Conclusion and Suggestions

In summary , it could be said that when the populations had five distributions on equal variance, all test statistics could control the type I error on almost sample sizes, both equal and unequal sizes. The test statistics with the highest testing power that could control type I errors was WBR test. In the case of unequal variance with the WBR test, B-M test, Y-W test, and Exact Wilcoxon signed-rank test could not control the type I error in almost all conditions especially when the population was enumerated log-normal distribution, gamma distribution and exponential distribution. However, when considering together both the capacity to control the type I error and highest testing power, the Welch was test is the best test statistic. Therefore, the application of the said test statistics must be carefully done in real cases.

References

Ahad, N. A., Othman, A. R., & Yahaya, S. S. (2011). Type I error rates of the two-sample pseudo-median procedure. *Journal of Modern Applied Statistical Methods*, *10,* 418-423.

Bradley, J. V. (1978). Robustness?. *Journal of Mathematical and Statistical Psychology*, *31,* 321-339.

Bridge, P. D., & Sawilowsky, S. S. (1999). Increasing physicians awareness of the impact of statistics on research outcomes: comparative power of the t-test and Wilcoxon rank-sum test in small samples applied research. *Journal of Clinical Epidemiology*, *52,* 229-235.

Brunner, E., & Munzel, U. (2000) The nonparametric behrens-fisher problem: Asymptotic theory and a small-sample approximation. *Biometrical Journal*, *42,* 17-25.

Cohen, J. (1988). *Statistical Power Analysis for the Behavioral Sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates.

Efron, B., & Tibshirani, R. J. (1993). *An Introduction to the Bootstrap.* New York: Chapman & Hall.

Fagerland, M. W. & Sandvik, L. (2009). Performance of five two-sample location tests for skewed distributions with unequal variances. *Contemporary Clinical Trial*, *30,* 490-496.

Harwell, M. R., Rubinstein, E. N., Hayes, W. S., & Olds, C. C. (1992). Summarizing Monte Carlo results in methodological research: The one- and two-factor fixed effects ANOVA cases. *Journal of Educational Statistics*, *17,* 315-339.

Harwell, M. R., & Serlin, R. C. (1989). A nonparametric test statistic for the general linear model. *Journal of Educational Statistics*, *14,* 351-371.

Mann, H. B., & Whitney, D. R. (1947). On a Test of Whether One of Two Random Variables is Stochastically Larger than the other. *Annals of Mathematical Statistics*, *18,* 50-60.

Medina, J. M., Kimberg, D. Y., Chatterjee, A., & Coslett, H. B. (2010). Inappropriate usage of the Brunner-Munzel test in recent voxel-based lesion-symtom mapping studies. *Neuropsychologia*, *48,* 341-343.

Reiczigel, J., Zakarias, I., & Rozsa, L. (2005). A bootstrap test of stochastic equality of two populations. *The American Statistician*, *59,* 1-6.

Stonehouse, J. M., & Forrester, G. J. (1998). Robustness of the t and U tests under combined assumption violations. *Journal of Applied Statistics*, *25,* 63-74.

Welch, B. L. (1938). The significance of the difference between two means when the population variancesare unequal. *Biometrika*, *29,* 350-362.

Wilcox, R. R. (1990). Comparing the mean of two independent group. *Biometrical Journal*, *32,* 771-780.

Wilcox, R. R. (1994). Some results on the Tukey-Mclaughlin and Yuen methods for trimmed means when distribution are skewed. *Biometrical Journal*, *3,* 259-273.

Wilcox, R. R. (2005). *Introduction to robust estimation and hypothesis testing* (2nd ed.). San Diego, CA: Academic Press.

Wilcox, R. R., & Keselman, H. J. (2003). Modern robust data analysis method: measures of central tendency. *Psychological Methods*, *8,* 254-274.

Wilcoxon, F. (1945). Individual Comparisons by Ranking Methods. *Biometrics*, *1,* 80-83.

Winter, J. C. F. (2013). Using the student's t-test with extremely small sample sizes. *Practical Assessment, Research & Evaluation, 18,* 1-12.

Yuen, K. K. (1974). The two-sample trimmed t for unequal population variances. *Biometrika*, *61,* 165-170.

Zimmerman, D. W., & Zumbo, B. D. (1993a). Rank transformations and the power of the Student t test and Welch t test non-normal populations with unequal variances. *Canadian Journal of Experimental Psychology*, *47,* 523-539.

Zimmerman, D. W., & Zumbo, B. D. (1993b). The relative power of parametric and nonparametric statistical methods. In G. Keren & C. Lewis (Eds.), *A handbook for data analysis in the behavioral sciences: Methodological issues* (pp. 481-517). Hillsdale, NJ: Erlbaum.