กรุปอินเวอร์สของเมทริกซ์ขนาด 3×3 เหนือริง

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The Group Inverse of 3×3 Matrices Over a Ring.

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Abstract

In this paper, we study conditions for the existence of the group inverse of the 3×3 matrix $N=\begin{bmatrix} e & h_1 & h_2 \\ v_1 & a & c \\ v_2 & b & d \end{bmatrix}$ over an arbitrary ring R with unity 1, when $M=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is the submatrix of N has the group inverse in $R^{2\times 2}$.

Keywords: von Neumann regularity, $\{1,2\}$-inverse, Group inverse, Matrix over a ring.

Introduction

Let $\mathbb{C}$ and $\mathbb{R}$ be the field of complex numbers and real numbers respectively. For a positive integers $m,n,$ let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ matrices over $\mathbb{C}$. The set of all complex vectors, or $n\times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$. We denote the identity and the zero matrix in $\mathbb{C}^{n\times n}$ by I_n and O_n, respectively. Note that A^* stands for $(\bar{A})^{\mathrm{T}}$.

For a given $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying

$$AXA=A, \tag{1}$$
$$XAX=X, \tag{2}$$
$$(AX)^*=AX, \tag{3}$$
$$(XA)^*=XA, \tag{4}$$

is called the *Moore-Penrose inverse* of A and is denoted by $A^{\dagger}$ (see Ben-Israel, & Greville, 2003).

We also consider the following equations which are applicable to square matrices

$$AX=XA. \tag{5}$$

For a given $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying (1), (2), and (5) is called the *group inverse* of A and denoted by $A^{\#}$.

Unlike the Moore-Penrose inverse, which always exists, the group inverse need not exist for all square matrices. A well known necessary and sufficient condition for the existence of $A^{\#}$ is that $\mathrm{rank}(A)=\mathrm{rank}(A^2)$. If A is nonsingular, then $A^{\#}=A^{-1}=A^{\dagger}$.

The group inverse has applications in singular differential and difference equations, Markov chains and iterative methods. Heinig, 1997, pp. 321-342 investigated the group inverse of Sylvester transformation. Wei, & Diao, 2005, pp. 109-123 studied the representation of the group inverse of a real singular Toeplitz matrix which arises in scientific computing and engineering. Catral, Olesky, &

Driessche, 2008, pp. 219-233 studied the existence of $A^{\#}$ (see Cao, Ge, Wang, & Zhang, 2013).

If A and B are square invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. However, for a generalized inverse this need not be true. Rajesh Kannan & Bapat, 2014 (Theorem 2.2) asserted that for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if $BB^{*}A^{\dagger}A$ and $A^{*}ABB^{\dagger}$ are Hermitian.

Preliminaries and Auxiliary Results

We shall be now concerned with generalized inverses that satisfy some, but not all, of the four Penrose equations.

Definition 2.1. [Ben-Israel & Greville, 2003, p. 40]. For any $A \in \mathbb{C}^{m\times n}$, let $A\{i,j,\ldots,k\}$ denote the set of the matrices $X \in \mathbb{C}^{n\times m}$ which satisfies equations (i), (j), …, (k) from among equations (1) - (4). A matrix $X \in A\{i,j,\ldots,k\}$ is called an $\{i,j,\ldots,k\}$- inverse of A, and also denoted by $A^{(i,j,\ldots,k)}$.

The examples are {1}- inverse (inner inverse), {1, 2}- inverse (reflexive inner inverse), {1, 3}- inverse (least squares inner inverse), {1, 4}- inverse (minimum norm inner inverse), {1, 2, 3}- inverse, {1, 2, 4}- inverse and {1, 2, 3, 4}- inverse, the last being the Moore-Penrose inverse of A.

Let R be a ring with unity 1. An element $a \in R$ is said to be von Neumann regular (regular) if there exists an element a^{-} of R such that $aa^{-}a = a$. In this case, a^{-} is called a {1}- inverse of a. An element a^{+} of R is a {1, 2}- inverse of a, which is given by $a^{+} = a^{-}aa^{-}$ (see [4] for example). Let R be a ring, not necessarily commutative.

Recall that an element $a \in R$ is said to be a unit if it has an inverse, if there is an element $a^{-1} \in R$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ (see for example Bhaskara Rao, 2002 (p.16). We denote an arbitrary {1}- inverse of A by A^{-} and {1, 2}- inverse of A, which is given by $A^{+} = A^{-}AA^{-}$. In the next section we will use the following result on regularity.

Lemma 2.2. *Let $A \in R^{3\times 3}$ be regular, $B \in R^{3\times 3}$, be such that there exists A^{+} such that $A^{+}B = B^{+}A = O_3$. If $(A+B)$ is regular then B is regular and $B^{-} = (A+B)^{-}$ is a {1}- inverse of B, for any $(A+B)^{-}$.*

Proof: Since $A^{+}B = B^{+}A = O_3$, then $B = (I_2 - AA^{+})(A+B) = (A+B)(I_2 - A^{+}A)$. Hence

$$B(A+B)^{-}B = (I_2 - AA^{+})(A+B)(A+B)^{-}(A+B)(I_2 - A^{+}A)$$
$$= (I_2 - AA^{+})(A+B)(I_2 - A^{+}A) = B.$$

Therefore, B is regular and $(A+B)^{-}$ is an {1}- inverse of B. □

Recently, Patricio, & Hartwig, 2010 characterize the existence of the group inverse of a two by two matrix with zero (2,2) entry, over an arbitrary ring. Castro-Gonzalez, Robles, & Velez-Cerrada, 2013 gave the conditions for the existence of the 2×2 the matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ has group inverse in $R^{2\times 2}$ in a ring with unity 1, and derived a representation of the group inverse of M in the case when either the entry a or d has a group inverse in R. Cao, et al., 2013 studied the group inverse of 2×2 block matrices $M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ over rings R with unity 1, where $CA = C$, $AB = B$, and the group inverse of $D - CB$ exists. In this paper, we study the group inverse of 3×3 matrices

$$N = \begin{bmatrix} e & h_1 & h_2 \\ v_1 & a & c \\ v_2 & b & d \end{bmatrix} =: \begin{bmatrix} e & h^{T} \\ v & M \end{bmatrix} \tag{2.1}$$

over an arbitrary ring R with unity 1, under the condition the 2×2 submatrix M has group inverse.

Now we assume that $M \in R^{2\times 2}$ is regular and that M^{+} is a fixed but arbitrary {1, 2}- inverse of M. Let us introduce the notation

$$E = I_2 - MM^{+},\ F = I_2 - M^{+}M,\ s = e - h^{T}M^{+}v. \tag{2.2}$$

We note that $FF = F$ and $EE = E$, since

$$
\begin{aligned}
EE = (I_2 - MM^+)(I_2 - MM^+) &= (I_2 - MM^+) - MM^+(I_2 - MM^+) \\
&= (I_2 - MM^+) - (MM^+ - MM^+MM^+) \\
&= I_2 - MM^+ - MM^+ + (MM^+M)M^+ \\
&= I_2 - MM^+ - MM^+ + MM^+ \\
&= I_2 - MM^+ = E,
\end{aligned}
$$

and

$$
\begin{aligned}
FF = (I_2 - M^+M)(I_2 - M^+M) &= (I_2 - M^+M) - M^+M(I_2 - M^+M) \\
&= I_2 - M^+M - (M^+M - M^+MM^+M) \\
&= I_2 - M^+M - M^+M + (M^+MM^+)M \\
&= I_2 - M^+M - M^+M + M^+M \\
&= I_2 - M^+M = F.
\end{aligned}
$$

We can decompose the matrix $N \in R^{3\times 3}$ as follows.

***Lemma* 2.3.** *The matrix N in (2.1) can be factored into*

$$
N = \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ M^+ v & I_2 \end{bmatrix} =: XZY, \quad (2.3)
$$

where

$$
X = \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix}, \; Z = \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 0^T \\ M^+ v & I_2 \end{bmatrix}.
$$

Proof: Consider

$$
\begin{aligned}
XZY &= \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ M^+ v & I_2 \end{bmatrix} \\
&= \begin{bmatrix} s + h^T M^+ Ev + h^T F M^+ v + h^T M^+ M M^+ v & h^T F + h^T M^+ M \\ Ev + MM^+ v & M \end{bmatrix} \\
&= \begin{bmatrix} (e - h^T M^+ v) + h^T M^+ (I_2 - MM^+) v + h^T (I_2 - M^+ M) M^+ v + h^T M^+ v & h^T F + h^T M^+ M \\ Ev + MM^+ v & M \end{bmatrix} \\
&= \begin{bmatrix} e & h^T \\ v & M \end{bmatrix} = N.
\end{aligned}
$$

□

We have a useful characterization the $N^{\#}$ as in [4, Lemma 1.2].

***Proposition* 2.4.** *Let $X, Y, Z \in R^{3\times 3}$. If $N = XZY$ where X and Y are units and Z is regular, then the group inverse of N exists if and only if $T = ZYX + I_2 - ZZ^-$ is a unit of $R^{3\times 3}$, independent of the choice of Z^-. Equivalently, $S = YXZ + I_2 - Z^-Z$ is a unit, in which case*

$$
N^{\#} = XT^{-2}ZY = XZS^{-2}Y.
$$

The group inverse of a 3×3 matrix over ring with unity 1.

In the notation (2.2), we assume both Ev and $h^T F$ to be regular elements in $R^{2\times 1}$ and $R^{1\times 2}$ respectively. Set

$$
x = 1 - (Ev)^+ Ev, \quad y = 1 - h^T F (h^T F)^+, \quad w = ysx, \qquad (3.1)
$$

for fixed but arbitrary $(Ev)^+$ and $(h^T F)^+$, and s defined in (2.2). By direct computation, we see that $xx = x$ and $yy = y$.

The von Neumann regularity of the matrix $Z \in R^{3\times 3}$ defined as in (2.3) is characterized in terms of the regularity of w as an element of the ring R in our next lemma. A representation for a {1}-inverse of A when it exists will prove extremely useful in the solution to our problem.

***Lemma* 3.1.** *Let E, F, s, x, y and w be as in (2.2) and (3.1). We have that $Z = \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix}$ is regular in $R^{3\times 3}$ if and only if w is regular in R.*

Proof: We must show that,

$$
EM = MF = O_2, \; FM^+ = M^+E = O_2, \; yh^T F = (h^T F)^+ y = O_2,
$$

where O_2 is the zero square matrix of order 2.

Firstly, from (2.2), $E = I_2 - MM^+$, $F = I_2 - M^+M$, we have

$$
\begin{aligned}
&EM = (I_2 - MM^+)M = M - MM^+M = O_2 = M - MM^+M = M(I_2 - M^+M) = MF, \\
&FM^+ = (I_2 - M^+M)M^+ = M^+ - M^+MM^+ = O_2 = M^+ - M^+MM^+ = M^+(I_2 - MM^+) = M^+E, \\
&yh^T F = (1 - h^T F (h^T F)^+) h^T F = h^T F - h^T F (h^T F)^+ h^T F = h^T F - h^T F = O_2, \\
&(h^T F)^+ y = (h^T F)^+ (1 - h^T F (h^T F)^+) = (h^T F)^+ - (h^T F)^+ h^T F (h^T F)^+ = (h^T F)^+ - (h^T F)^+ = O_2.
\end{aligned}
$$

Now, consider

$$\begin{bmatrix} 1 & -ys(Ev)^+E \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} s & h^TF \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix} = \begin{bmatrix} s-ys(Ev)^+EEv & h^TF-ys(Ev)^+EM \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} s-ys(Ev)^+Ev-h^TF(h^TF)^+s+ys(Ev)^+0F(h^TF)^+s & h^TF-ys(Ev)^+0 \\ Ev-0(h^TF)^+s & M \end{bmatrix}$$

$$= \begin{bmatrix} s-(1-h^TF(h^TF)^+)s(Ev)^+Ev-h^TF(h^TF)^+s & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} s-s(Ev)^+Ev+h^TF(h^TF)^+s(Ev)^+Ev-h^TF(h^TF)^+s & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} s\left[1-(Ev)^+Ev\right]-h^TF(h^TF)^+s\left[1-(Ev)^+Ev\right] & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} sx-h^TF(h^TF)^+sx & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} \left[1-h^TF(h^TF)^+\right]sx & h^TF \\ Ev & M \end{bmatrix}$$

$$= \begin{bmatrix} ysx & h^TF \\ Ev & M \end{bmatrix}.$$

Let us denote

$$PZQ := \begin{bmatrix} 1 & -ys(Ev)^+E \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} s & h^TF \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix}$$
$$= \begin{bmatrix} ysx & h^TF \\ Ev & M \end{bmatrix} = \begin{bmatrix} w & h^TF \\ Ev & M \end{bmatrix} =: \tilde{Z}. \tag{3.2}$$

From above $PZQ=\tilde{Z}$, consider P and Q are nonsingular matrices. We have

$$\tilde{Z} = PZQ = P(ZZ^-Z)Q = PZ(QQ^{-1})Z^{-1}(P^{-1}P)ZQ = PZQ(Q^{-1}Z^{-1}P^{-1})PZQ = \tilde{Z}(Q^{-1}Z^{-1}P^{-1})\tilde{Z}.$$

Therefore Z is regular if and only if $\tilde{Z}$ is regular.

Using $w=ysx$ from (3.1), we can write

$$\tilde{Z} = \begin{bmatrix} 0 & h^TF \\ Ev & M \end{bmatrix} + \begin{bmatrix} w & 0^T \\ 0 & O_2 \end{bmatrix} = H + W.$$

Next, we must show that $\tilde{Z}$ is regular if and only if w is regular. We have that H is regular, which

$$H^- = \begin{bmatrix} 0 & 0^T \\ -F(h^TF)^+s & I_2 \end{bmatrix}\begin{bmatrix} xw^-y & (Ev)^+E \\ -F(h^TF)^+ & M^+ \end{bmatrix}\begin{bmatrix} 0 & -ys(Ev)^+E \\ 0 & I_2 \end{bmatrix}$$

is a {1}- inverse of Z, and

$$H^+ = \begin{bmatrix} 0 & (EM)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix}$$

is a {1, 2}- inverse of H such that $WH^+ = H^+W = O_3$, since

$$WH^+ = \begin{bmatrix} w & 0^T \\ 0 & O_2 \end{bmatrix}\begin{bmatrix} 0 & (EM)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix} = \begin{bmatrix} 0 & w(EM)^+E \\ 0 & O_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & w(EM)^+(I_2-MM^+) \\ 0 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & wE^+M^+ \\ 0 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0^T \\ 0 & O_2 \end{bmatrix},$$

and

$$H^+W = \begin{bmatrix} 0 & (EM)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix}\begin{bmatrix} w & 0^T \\ 0 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0^T \\ F(h^TF)^+w & O_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0^T \\ F(h^TF)^+ysx & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0^T \\ 0 & O_2 \end{bmatrix}.$$

Now, if $\tilde{Z}$ is regular, then W is regular, by Lemma 2.2. This implies that w is regular.

Conversely, assume that w is regular. Let

$$X = \begin{bmatrix} xw^-y & (Ev)^+E \\ F(h^TF)^+ & M^+ \end{bmatrix}.$$

We claim that X is a $\{1\}$- inverse of $\tilde{Z}$ Consider,

$$\tilde{Z}\, X\, \tilde{Z} = \begin{bmatrix} w & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} xw^- w + 1 - x & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix} = \tilde{Z}.$$

Now,

$$\begin{aligned} X\tilde{Z} &= \begin{bmatrix} xw^- y & (Ev)^+ E \\ F(h^T F)^+ & M^+ \end{bmatrix} \begin{bmatrix} ysx & h^T F \\ Ev & M \end{bmatrix} \\ &= \begin{bmatrix} xw^- yysx + (Ev)^+ EEv & xw^- yh^T F + (Ev)^+ EM \\ F(h^T F)^+ ysx + M^+ Ev & F(h^T F)^+ h^T F + M^+ M \end{bmatrix} \\ &= \begin{bmatrix} xw^- ysx + (Ev)^+ Ev & 0^T \\ 0 & F(h^T F)^+ h^T F + M^+ M \end{bmatrix} \\ &= \begin{bmatrix} xw^- w + (Ev)^+ Ev & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix} \\ &= \begin{bmatrix} xw^- w + 1 - x & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix}. \end{aligned}$$

Indeed,

$$\begin{aligned} \tilde{Z}X\tilde{Z} &= \begin{bmatrix} ysx & h^T F \\ Ev & M \end{bmatrix} \begin{bmatrix} xw^- w + 1 - x & 0^T \\ 0 & M^+ M + F(h^T F)^+ h^T F \end{bmatrix} = \tilde{Z} \\ &= \begin{bmatrix} ysx(xw^- w + 1 - x) & h^T F(M^+ M + F(h^T F)^+ h^T F) \\ Ev(xw^- w + 1 - x) & M(M^+ M + F(h^T F)^+ h^T F) \end{bmatrix} \\ &= \begin{bmatrix} ysxxw^- w + ysx - ysxx & h^T FM^+ M + h^T FF(h^T F)^+ h^T F \\ Evxw^- w + Ev - Evx & MM^+ M + MF(h^T F)^+ h^T F \end{bmatrix} \\ &= \begin{bmatrix} ysxw^- w + ysx - ysx & h^T F(h^T F)^+ h^T F \\ Ev(1 - (Ev)^+ Ev)w^- w - Ev(1 - (Ev)^+ Ev) + Ev & MM^+ M \end{bmatrix} \\ &= \begin{bmatrix} ysxw^- w & h^T F \\ Evw^- w - Ev(Ev)^+ Evw^- w - Ev + Ev(Ev)^+ Ev + Ev & M \end{bmatrix} \\ &= \begin{bmatrix} ww^- w & h^T F \\ Evw^- w - Evw^- w - Ev + Ev + Ev & M \end{bmatrix} \\ &= \begin{bmatrix} w & h^T F \\ Ev & M \end{bmatrix} \\ &= \begin{bmatrix} ysx & h^T F \\ Ev & M \end{bmatrix}. \end{aligned}$$

In view of (3.2) we conclude that a $\{1\}$- inverse of Z is given by (3.3) and, thus, Z is regular. It remains to prove (3.4) but the proof of this is straightforward. In fact, for Z in (2.3) we have

$$Z^- = \begin{bmatrix} 1 & 0^T \\ -F(h^T F)^+ s & I_2 \end{bmatrix} \begin{bmatrix} xw^- y & (Ev)^+ E \\ -F(h^T F)^+ & M^+ \end{bmatrix} \begin{bmatrix} 1 & -ys(Ev)^+ E \\ 0 & I_2 \end{bmatrix} \tag{3.3}$$

is a $\{1\}$- inverse of Z. By direct computation we have

$$I_3 - ZZ^- = \begin{bmatrix} (1 - ww^-)y & -(1 - ww^-)ys(Ev)^+ E \\ 0 & (1 - Ev(Ev)^+)E \end{bmatrix}. \tag{3.4}$$

From, our assumption, we have M is the group inverse. Then we can set $M^{+}=M^{\#}$. In this case, in the notation of (2.2) we have $E=I_2-MM^{\#}=I_2-M^{\#}M=F$. It follows that $ME=EM^{\#}=O_2$, since

$$\begin{aligned} ME &= M(I_2-M^{\#}M) = M-MM^{\#}M = M-M = O_2,\\ EM^{\#} &= (I_2-M^{\#}M)M^{\#} = M^{\#}-M^{\#}MM^{\#} = M^{\#}-M^{\#} = O_2. \end{aligned}$$

We claim that $M+E$ is a unit of $R^{2\times 2}$ and

$$(M+E)^{-1}=M^{\#}+E.$$

For,

$$\begin{aligned} (M+E)(M^{\#}+E) &= M(M^{\#}+E)+E(M^{\#}+E)\\ &= MM^{\#}+ME+EM^{\#}+EE\\ &= MM^{\#}+O_2+O_2+E\\ &= MM^{\#}+(I_2-MM^{\#})\\ &= O_2, \end{aligned}$$

and

$$\begin{aligned} (M^{\#}+E)(M+E) &= M^{\#}(M+E)+E(M+E)\\ &= M^{\#}M+M^{\#}E+EM+EE\\ &= M^{\#}M+O_2+O_2+E\\ &= M^{\#}M+(I_2-M^{\#}M)\\ &= O_2. \end{aligned}$$

$\square$

***Theorem* 3.2.** *Let M be group invertible. With the notation (2.2) and under the assumptions of (3.1), with M^{+} replaced by $M^{\#}$ if w is regular in R then the group inverse of the matrix*

$$N=\begin{bmatrix} e & h_1 & h_2\\ v_1 & a & c\\ v_2 & b & d \end{bmatrix}=\begin{bmatrix} e & h^T\\ v & M \end{bmatrix}$$

exists if and only if

$$r=sx-h^TEv+(1-ww^{-})y[s+(1+h^T(M^{\#})^2v)x] \quad (3.5)$$

is a unit of R. In this case,

$$N^{\#}=\begin{bmatrix} \omega_1 & \Omega_3\\ \Omega_2 & \Omega_4 \end{bmatrix},$$

where

$$\begin{aligned} \omega_1 &= \gamma-x(\lambda\gamma-r^{-1}),\\ \Omega_2 &= -M^{\#}(v-M^{\#}vx)\gamma+(M^{\#}vx+Ev)(\lambda\gamma-r^{-1}),\\ \Omega_3 &= -\Theta+x(\gamma h^T(M^{\#})^2+\lambda\Theta-r^{-1}h^TM^{\#}),\\ \Omega_4 &= M^{\#}+M^{\#}(v-M^{\#}vx)\Theta-(M^{\#}vx+Ev)(\gamma h^T(M^{\#})^2+\lambda\Theta-r^{1}h^TM^{\#}), \end{aligned} \quad (3.6)$$

with

$$\begin{aligned} \gamma &= r^{-1}(1-ww^{-})y,\\ \Theta &= \gamma \mathrm{h}^T M^{\#}+r^{-1}h^TE,\\ \lambda &= \gamma(1+h^T(M^{\#})^2(v-M^{\#}vx))+r^{-1}(s+(1+h^T(M^{\#})^2v)x). \end{aligned} \quad (3.7)$$

Proof: Write $N=XZY$ as in (2.3), by Proposition 2.4, the group inverse of N exists if and only if $T=ZYX+I_3-ZZ^{-}$ is a unit, independent of the choice of Z^{-}. For the {1}-inverse provided in Lemma 3.1, we have

$$T=ZYX+I_3-ZZ^{-}=\begin{bmatrix} s+(1-ww^{-})y & sh^TM^{\#}+h^TE-(1-ww^{-})ys(Ev)^{+}E\\ v & M+vh^TM^{\#}+(1-Ev(Ev)^{+})E \end{bmatrix}.$$

Now, let us introduce the matrix

$$G=\begin{bmatrix} 1 & (Ev)^{+}E-h^TM^{\#}\\ 0 & I_2 \end{bmatrix}.$$

We have

$$TG = \begin{bmatrix} s+(1-ww^-)y & u^T \\ v & K \end{bmatrix}, \tag{3.8}$$

where

$$\begin{aligned} K &= M+E+MM^{\#}v(Ev)^{+}E = (M+E)(1+M^{\#}v(Ev)^{+}E), \\ u &= (s+(1-ww^-)y)((Ev)^{+}E - h^T M^{\#}) + (sh^T M^{\#} + h^T E - (1-ww^-)ys(Ev)^{+}E). \end{aligned} \tag{3.9}$$

Since the element $M^{\#}v(Ev)^{+}E$ is 2-nilpotent, because

$$(M^{\#}v(Ev)^{+}E)(M^{\#}v(Ev)^{+}E) = M^{\#}v(Ev)^{+}EM^{\#}v(Ev)^{+}E = M^{\#}v(Ev)^{+}0v(Ev)^{+}E = O.$$

It follows that $1+M^{\#}v(Ev)^{+}E$ is a unit of R.

$$\begin{aligned} (1+M^{\#}v(Ev)^{+}E)(1-M^{\#}v(Ev)^{+}E) &= ((1-M^{\#}v(Ev)^{+}E) + M^{\#}v(Ev)^{+}E(1-M^{\#}v(Ev)^{+}E)) \\ &= 1-M^{\#}v(Ev)^{+}E + M^{\#}v(Ev)^{+}E - M^{\#}v(Ev)^{+}EM^{\#}v(Ev)^{+}E \\ &= 1-M^{\#}v(Ev)^{+}0v(Ev)^{+}E \\ &= 1. \end{aligned}$$

Moreover $M+E$ is a unit because M has group inverse. Thus, K is a unit and

$$S = K^{-1} = (1-M^{\#}v(Ev)^{+}E)(M^{\#}+E). \tag{3.10}$$

On account that the element $(2,2)$ of the matrix TG is a unit, it follows that TG is a unit of $R^{2\times 2}$ if and only if the Schur complement is a unit of R. Therefore, the matrix T is a unit if and only if

$$r = s+(1-ww^-)y - uSv \tag{3.11}$$

is a unit in R. From (3.10), we get $Sv = Ev + M^{\#}vx.$

Now,

$$\begin{aligned} Sv &= [(1-M^{\#}v(Ev)^{+}E)(M^{\#}+E)]v \\ &= [(M^{\#}+E) - M^{\#}v(Ev)^{+}E(M^{\#}+E)]v \\ &= [M^{\#}+E - M^{\#}v(Ev)^{+}EM^{\#} - M^{\#}v(Ev)^{+}EE]v \\ &= M^{\#}v + Ev - M^{\#}v(Ev)^{+}0v - M^{\#}v(Ev)^{+}EEv \\ &= Ev + M^{\#}v - M^{\#}v(Ev)^{+}EEv \\ &= Ev + M^{\#}v(1-(Ev)^{+}Ev) \\ &= Ev + M^{\#}vx. \end{aligned}$$

Further, using that last relation of (3.9) we obtain

$$uSv = s(1-x) + h^T Ev - (1-ww^-)y[(s-1) + (1+h^T(M^{\#})^2 v)x].$$

By substituting this expression in (3.11), we conclude that r has the form given in (3.5). By Proposition 2.4, $N^{\#} = XT^{-2}ZY$. From (3.8), it follows that

$$(TG)^{-1} = \begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}.$$

Next, we compute

$$XT^{-1} = XG(TG)^{-1} = \begin{bmatrix} 1 & h^T M^\# \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (Ev)^+ E - h^T M^\# \\ 0 & 1 \end{bmatrix} (TG)^{-1}$$
$$= \begin{bmatrix} xr^{-1} & (Ev)^+ E - xr^{-1}uS \\ -Sr^{-1} & S + Svr^{-1}uS \end{bmatrix}. \qquad (3.12)$$

Now,

$$XT^{-1} = XG(TG)^{-1} = \begin{bmatrix} 1 & h^T M^\# \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (Ev)^+ E - h^T M^\# \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$
$$= \begin{bmatrix} 1 & (Ev)^+ E - h^T M^\# + h^T M^\# \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$
$$= \begin{bmatrix} 1 & (Ev)^+ E \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$
$$= \begin{bmatrix} r^{-1} - (Ev)^+ ESvr^{-1} & -r^{-1}uS + (Ev)^+ E(S + Svr^{-1}uS) \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$
$$= \begin{bmatrix} r^{-1} - (Ev)^+ ESvr^{-1} & -r^{-1}uS + (Ev)^+ ES + (Ev)^+ ESvr^{-1}uS \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$
$$= \begin{bmatrix} r^{-1} - (Ev)^+ Evr^{-1} & (Ev)^+ E - r^{-1}uS + (Ev)^+ Evr^{-1}uS \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}; ES = E$$
$$= \begin{bmatrix} (1 - (Ev)^+ Ev)r^{-1} & (Ev)^+ E - (1 - (Ev)^+ Ev)r^{-1}uS \\ 0 - Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$
$$= \begin{bmatrix} xr^{-1} & (Ev)^+ E - xr^{-1}uS \\ -Svr^{-1} & S + Svr^{-1}uS \end{bmatrix}$$

the last equality is due to the fact that $ES = E.$ In the sequel, we denote

$$\gamma = r^{-1}(1 - ww^-)y \quad \text{and} \quad \Theta = \gamma h^T M^\# + r^{-1} h^T E.$$

From (3.10), (3.9), and (3.11) it follows that

$$\begin{aligned} SM &= M^\# M, \\ r^{-1}uM^\# M &= -\gamma h^T M^\#, \\ r^{-1}uSv &= r^{-1}(s + (1 - ww^-)y) - 1 = r^{-1}s + \gamma - 1 \end{aligned} \qquad (3.13)$$

respectively. In deriving the last equality, we have multiplied on the left expression (3.11) by r^{-1}. Then

$$
\begin{aligned}
T^{-1}ZY &= G(TG)^{-1}ZY \\
&= G(TG)^{-1}\begin{bmatrix} s & h^T E \\ Ev & M \end{bmatrix}\begin{bmatrix} 1 & 0 \\ M^{\#}v & I_2 \end{bmatrix} \\
&= G\begin{bmatrix} r^{-1} & -r^{-1}uS \\ -Svr^{-1} & S+Svr^{-1}uS \end{bmatrix}\begin{bmatrix} s & h^T E \\ v & M \end{bmatrix} \\
&= \begin{bmatrix} 1 & (Ev)^{+}E - h^T M^{\#} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-\gamma & \Theta \\ Sv\gamma & M^{\#}M - Sv\Theta \end{bmatrix} \\
&= \begin{bmatrix} 1-(1+h^T(M^{\#})^2 v)x\gamma & -h^T M^{\#} + (1+h^T(M^{\#})^2 v)x\Theta \\ Sv\gamma & M^{\#}M - Sv\Theta \end{bmatrix}.
\end{aligned}
\tag{3.14}
$$

Using (3.12) and (3.14), we obtain

$$
\begin{aligned}
XT^{-1}ZY &= \begin{bmatrix} xr^{-1} & (Ev)^{+}E - xr^{-1}uS \\ -Svr^{-1} & S+Svr^{-1}uS \end{bmatrix}\begin{bmatrix} 1-(1+h^T(M^{\#})^2 v)x\gamma & -h^T M^{\#} + (1+h^T(M^{\#})^2 v)x\Theta \\ Sv\gamma & M^{\#}M - Sv\Theta \end{bmatrix} \\
&= \begin{bmatrix} \omega_1 & \Omega_3 \\ \Omega_2 & \Omega_4 \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1 &= (1-x)\gamma - xr^{-1}(uS^2 v\gamma - 1 + (1+h^T(M^{\#})^2 v)x\gamma), \\
\Omega_2 &= S^2 v\gamma + uvr^{-1}(uS^2 v\gamma - 1 + (1+h^T(M^{\#})^2 v)x\gamma), \\
\Omega_3 &= (x-1)\Theta + xr^{-1}(uS^2 v\Theta - uM^{\#} - h^T M^{\#} + (1+h^T(M^{\#})^2 v)x\Theta), \\
\Omega_4 &= M^{\#} - S^2 v\Theta - Svr^{-1}(uS^2 v\Theta - uM^{\#} - h^T M^{\#} + (1+h^T(M^{\#})^2 v)x\Theta).
\end{aligned}
\tag{3.15}
$$

From (3.10), it follows that $S^2 v = Sv - M^{\#}(v - M^{\#}vx)$.

$$
\begin{aligned}
S^2 v = S(Sv) &= S(Ev + M^{\#}vx) \\
&= S(E)v + SM^{\#}vx \\
&= S(1 - M^{\#}M)v + SM^{\#}vx \\
&= Sv - SM^{\#}Mv + SM^{\#}vx \\
&= Sv - M^{\#}v + ((1 - M^{\#}v(Ev)^{+}E)(M^{\#} + E))M^{\#}vx \\
&= Sv - M^{\#}v + ((M^{\#}M^{\#}vx + EM^{\#}vx) - M^{\#}v(Ev)^{+}EEM^{\#}vx) \\
&= Sv - M^{\#}v + M^{\#}M^{\#}vx + EM^{\#}vx - M^{\#}v(Ev)^{+}EEM^{\#}vx \\
&= Sv - M^{\#}v + M^{\#}M^{\#}vx \\
&= Sv - M^{\#}(v - M^{\#}vx).
\end{aligned}
$$

Using this latter expression and (3.13) we get

$$
r^{-1}uS^2 v = r^{-1}uSv - r^{-1}uM^{\#}(v - M^{\#}vx) = r^{-1}s + \gamma - 1 + \gamma h^T(M^{\#})^2(v - M^{\#}vx).
$$

By substituting this into (3.15), using $r^{-1}uM^{\#} = -\gamma h^T(M^{\#})^2$, and regrouping terms we get (3.6).

□

Conclusions

In this work we have studied conditions for the existence of the group inverse of the 3×3 matrix over a ring with unity 1 under some development certain conditions.

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