Generalized Continuous Functions from Any Topological Space into Product

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Summary

In this paper we prove some relationship between continuous functions and g-continuous functions and study g-continuous functions from any topological spaces into product spaces. In general continuous functions is g-continuous functions but not conversely. We can give some characterizations of this type of continuity.

Introduction

Balachandran, Sundaram and Maki (1991) introduced the notion of generalized continuous (g-continuous) functions by using g-closed sets and obtained some of their properties. The purpose of this paper is to present the proof of some relationship between continuous functions and g-continuous functions and some properties of g-continuous functions from any topological space into product space.

Preliminaries

The symbols X, Y and Z denote topological spaces with no separation axioms assumed unless explicitly stated. The closure and the interior of a subset A of a space X are denoted by $\overline{A}$ and $Int(A)$, respectively.

Definition 1. A subset A of a space X is said to be generalized closed (g-closed) provided that $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open in X. A subset A is said to be generalized open(g-open) if its complement is generalized closed (Levine, 1970).

Theorem 1. *A set A is g-open if and only if* $F \subseteq Int(A)$ *whenever F is closed and* $F \subseteq A$ (Levine, 1970).

Theorem 2. *Let* $\{Y_\alpha \mid \alpha \in I\}$ *be any family of topological spaces, and* $f : X \rightarrow \prod_{\alpha \in I} Y_\alpha$ *a map. Then f is continuous if and only if* $\pi_\alpha \circ f$ *is* *continuous for each* $\alpha \in I$ (Dugunji,1966).

Theorem 3. *Let* $\{X_\alpha | \alpha \in I\}$ *and* $\{Y_\alpha | \alpha \in I\}$ *be any family of topological spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a map. Define* $\prod f_\alpha : \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ *by* $(x_\alpha)_{\alpha\in I} \to (f_\alpha(x_\alpha))_{\alpha\in I}$.

Then : (i) If each f_α *is continuous ,then so is* $\prod f_\alpha$.

(ii) If each f_α *is an open map, and all but at most finitely many are surjective, then* $\prod f_\alpha$ *is also an open map.*

Relation between continuous functions and g-continuous functions

Definition 2. A map $f : X \to Y$ from a topological spaces X into a topological space Y is call g-continuous if the inverse image of closed set in Y is g-closed in X.

Theorem 4. *Let X and Y be topological spaces. Then* $f : X \to Y$ *is g-continuous if and only if the inverse image of every open set in Y is g-open in X.*

Proof. $(\rightarrow)$ Let G be any open set in Y. Then $Y - G$ is closed in Y. Since f is g-continuous, $f^{-1}(Y - G)$ is g-closed in X. But $f^{-1}(Y - G) = X - f^{-1}(G)$. This implies that $f^{-1}(G)$ is g-open in X.

$(\leftarrow)$ Let F be any closed set in Y. Then $Y - G$ is open in Y. By assumtion, $f^{-1}(Y - F)$ is g-open in X. But $f^{-1}(Y - F) = X - f^{-1}(F)$. This implies that $f^{-1}(F)$ is g-closed in X. Hence f is g-continuous.

Theorem 5. *Let X and Y be topological spaces. If* $f : X \to Y$ *is continuous, then f is g-continuous, but not conversely.*

Proof. Let F be a closed set in Y. Since f is continuous, $f^{-1}(F)$ is closed in X which implies that $f^{-1}(F)$ is g-closed in X. Hence f is g-continuous. The converse is not true as will be seen in the following example.

Example 1. Let $X = \{a,b.c\}$, $\Im = \{\phi,\{a\},X\}$, $Y = \{p,q\}$ and $\Im' = \{\phi,\{q\},Y\}$. Let $f : (X,\Im) \to (Y,\Im')$ be defined by $f(a) = f(c) = q$ and $f(b) = p$. It is easy to see that f is g-continuous but not continuous. Since $\{q\}$ is open in Y but

$f^{-1}(\{q\}) = \{a,c\}$ is not open in X.

Definition 3. A topological space X is called a $T_{\frac{1}{2}}$-space if every g-closed set in X is closed in X.

Theorem 6. *Let X be a $T_{\frac{1}{2}}$-space and Y a topological space. Then $f : X \to Y$ is continuous if and only if f is g-continuous.*

Proof. $(\to)$ By Theorem 5.

$(\leftarrow)$ Let F be a closed set in Y. Since f is g-continuous, $f^{-1}(F)$ is g-closed in X. Since X is a $T_{\frac{1}{2}}$-space, we have $f^{-1}(F)$ is closed in X.

Theorem 7. *Let X,Y and Z be topological spaces. If $f : X \to Y$ is g-continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is g-continuous.*

Proof. Let F be a closed set in Z. Since g is continuous, $g^{-1}(F)$ is closed in Y. Since f is g-continuous, we have $f^{-1}(g^{-1}(F))$ is g-closed in X. Hence $g \circ f$ is g-continuous.

Example 2. Let

$X = Y = Z = \{a,b,c\}, \Im = \{\phi,\{a,b\},X\}, \Im' = \{\phi,\{a\},\{b,c\},Y\}$ and $\Im'' = \{\phi,\{a,c\},Z\}$. Let $f : (X,\Im) \to (Y,\Im')$ be defined by $f(a) = c$, $f(b) = b$ and $f(c) = c$. Let $g : (Y,\Im') \to (Z,\Im'')$ be the identity map. It is easy to see that both f and g are g-continuous but $g \circ f$ is not g-continuous because $(g \circ f)^{-1}(\{b\}) = \{b\}$ is not g-closed in X.

Theorem 8. *Let X and Z be topological spaces and Y be a $T_{\frac{1}{2}}$-space. If $f : X \to Y$ and $g : Y \to Z$ are g-continuous, then $g \circ f$ is g-continuous.*

Proof. Let F be any closed set in Z. Since g is g-continuous, $g^{-1}(F)$ is g-closed in Y. But Y is $T_{\frac{1}{2}}$-space, so $g^{-1}(F)$ is closed in Y. Since f is g-continuous, it implies that $f^{-1}(g^{-1}(F))$ is g-closed in X. Hence $g \circ f$ is g-continuous.

Generalized continuous functions from any topological space into product space

In this section, we study g-continuous functions from any topological space into product space. We give some characterizations of this type of continuity.

Theorem 9. *Let* Y *be a topological space and let* $\{X_a | a \in I\}$ *be a family oftopological spaces.Let* $f: Y \to \prod_{\alpha \in I} X_\alpha$ *be a function. If* f *is g-continuous, then the composite function* $\pi_\alpha \circ f : Y \to X_\alpha$ *is g-continuous for each* $\alpha \in I$.

Proof. Suppose that f is g-continuous. Since $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$ is continuous for each $\alpha \in I$, it follows from Theorem 7 that $\pi_\alpha \circ f$ is g-continuous for each $\alpha \in I$.

The converse of Theorem 9 is not true as shown in the following example.

Example 3. Let

$X = \{1,2,3,4\}$, $\Im_x = \{\phi, \{1\}, \{2\}, \{1,2\}, X\}$, $Y_1 = Y_2 = \{a,b\}$,

$\Im_{Y1} = \{\phi, \{a\}, Y_1\}$, $\Im_{Y2} = \{\phi, \{a\}, Y_2\}$, $Y = Y_1 \times Y_2 = \{(a,a),(a,b),(b,a),(b,b)\}$ and

$\Im_Y = \{\phi, Y_1 \times Y_2, \{(a,a),(a,b)\}, \{(a,a),(b,a)\}, \{(a,a)\}, \{(a,a),(a,b),(b,a)\}\}$.

Define $f : X \to Y$ by $f(1) = (a,a)$, $f(2) = (b,b)$, $f(3) = (a,b)$ and $f(4) = (b,a)$. It is easy to see that $\pi_1 \circ f$ and $\pi_2 \circ f$ are g-continuous. However, $\{(b,b)\}$ is closed in Y but $f^{-1}(\{(b,b)\}) = \{2\}$ is not g-closed in X. Therefore f is not g-continuous.

Theorem 10. *Let* Y *be a* $T_{\frac{1}{2}}$ *-space and let* $\{X_\alpha | \alpha \in I\}$ *be a family of topological spaces.*

Let $f : Y \to \prod_{\alpha \in I} X_\alpha$ *be a function. Then* f *is g-continuous if and only if the composite function* $\pi_\alpha \circ f : Y \to X_\alpha$ *is g-continuous for each* $\alpha \in I$.

Proof. $(\rightarrow)$ By Theorem 9.

$(\leftarrow)$ Since Y is $T_{\frac{1}{2}}$ -space and by Theorem 6, $\pi_\alpha \circ f$ is continuous for each $\alpha \in I$. By Theorem 2, f is continuous. Hence f is g-continuous.

Corollary 1. *Let* $\{X_\alpha | \alpha \in I\}$ *and* $\{Y_\alpha | \alpha \in I\}$ *be families of topological spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a function and* $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_a$ *be* *defined by* $f((x_\alpha)_{\alpha\in I}) = (fa(x_\alpha))_{\alpha\in I}$. *If f is g-continuous, then* f_α *is g-continuous for each* $\alpha \in I$.

Proof. Let $\beta \in I$, and π_β and π'_β be the projections of $\prod_{\alpha\in I} X_\alpha$ and $\prod_{\alpha\in I} Y_\alpha$ onto X_β and Y_β respectively. By Theorem 7, $\pi'_\beta \circ f$ is g-continuous for each $\beta \in I$. Since $\pi'_\beta \circ f = f_\beta \circ \pi$ we have that $f_\beta \circ \pi_\beta$ is g-continuous. Let F be closed in Y_β. Then $\pi_\beta^{-1}(f_\beta(F)) = f_\beta(F) \times \prod_{\alpha\neq\beta} X_\alpha$ is g-closed in $\prod_{\alpha\in I} X_\alpha$. Therefore $f_\beta^{-1}(F)$ is g-closed in X_β. Hence f_β is g-continuous.

Corollary 2. *Let* $\{X_\alpha | \alpha \in I\}$ *be a family of* $T_{\frac{1}{2}}$*-spaces and* $\{Y_\alpha | \alpha \in I\}$ *a family of topological spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \rightarrow Y_\alpha$ *be a function and* $f: \prod_{\alpha\in I} X_\alpha \rightarrow \prod_{\alpha\in I} Y_\alpha$ *defined by* $f((x_\alpha)_{\alpha\in I} = (f_\alpha(x_\alpha))_{\alpha\in I}$. $f\left(\left(x_\alpha\right)_{\alpha\in I}\right) = \left(f_\alpha\left(x_\alpha\right)\right)_{\alpha\in I}$. *Then f is g-continuous if and only if* f_a *is g-continuous for each* $a \in I$.

Proof. $(\rightarrow)$ By Corollary 1.

$(\leftarrow)$ Suppose that f_α is g-continuous for each $\alpha \in I$. Since X_α is a $T_{\frac{1}{2}}$-space for each $\alpha \in I$, it follows from Theorem 6 that f_α is continuous for each $\alpha \in I$. Therefore by Theorem 3(i), f is continuous. Hence f is g-continuous.

Conclusion

This paper we obtain the main objectives as the following:

1. We obtain relations among continuous and g-continuous functions, they are:

1.1 If X is a $T_{\frac{1}{2}}$-space and Y is a topological space, then $f: X \rightarrow Y$ is continuous if and only if F is g-continuous.

1.2 If X and Z are topological spaces and Y is a $T_{\frac{1}{2}}$-space, then if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are g-continuous, then $g \circ f$ is g-continuous.

2. We prove some properties of g-continuous functions from any topological space into a product space.

The result are the following:

2.1 If Y is a $T_{\frac{1}{2}}$-space and $\{X_\alpha | \alpha \in I\}$ is a family of topological spaces, then a function $f: Y \rightarrow \prod_{\alpha\in I} X_\alpha$ is g-continuous if and only if the composite function $\pi_\alpha \circ f : Y \to X_\alpha$ is g-continuous for each $\alpha \in I$.

2.2 If $\{X_\alpha | \alpha \in I\}$ is a family of $T_{\frac{1}{2}}$ -spaces and $\{Y_\alpha | \alpha \in I\}$ is a family of topological spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be defined by
$f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then f is g-continuous if and only if f_α is g-continuous for each $\alpha \in I$.

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