Preservation Theorems Concerning g-Hausdorff and rg-Hausdorff Spaces

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Summary

In this paper we introduce and study the concepts of topological spaces called g-Hausdorff and rg-Hausdorff spaces and investigate preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces.

Introduction

Levin (1970) introduced the concept of generalized closed sets in topological space. Dunham and Levin (1980) further studied some properties of generalized closed sets. In (1991) Balachandran *et al* introduced the notion of g-continuous maps by using g-closed sets and obtained some of their properties. Further they have investigated the concept of gc-irresolute, strongly g-continuous and perfectly g-continuous maps. Palaniappan and Rao (1993) introduced the notion of regular generalized continuous maps, rg-irresoluteness. In the present paper, we introduce g-Hausdorff and rg-Hausdorff spaces and investigate preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces.

Preliminaries

Given any subset A in a topological space X, the closure and the interior of A are denoted by $\bar{A}$ and $Int\,(A)$, respectively.

Definition 1. A subset A of a topological space X is said to be regular open if $A = Int\,(\bar{A})$

Definition 2. A topological space X is a T_2-space (Hausdorff) if whenever x and y are distinct points of X there are disjoint open sets U and V with $x \in U$ and $y \in V$.

Definition 3. A subset A of a space X is said to be g-closed (resp. rg-closed) provided that $\bar{A} \subseteq U$ whenever $A \subseteq U$ and U is open (resp. regular open) in X. A subset A is said to be g-open (resp. rg-open) if its complement is g-closed (resp. rg-closed).

Definition 4. A funcion $f : X \rightarrow Y$ is said to be g-continuous (resp. rg-continuous) provided that for every closed subset F of Y, $f^{-1}(F)$ is g-closed (resp. rg-closed).

Definition 5. A funcion $f : X \rightarrow Y$ is said to be gc-irresolute (resp. rg-irresolute) provided that for every g-closed (resp. rg-closed) subset F of Y, $f^{-1}(F)$ is g-closed (resp. rg-closed).

Preservation Theorems Concerning g-Hausdorff and rg-Hausdorff Spaces

In this section, we define new concepts of g-Hausdorff and rg-Hausdorff spaces and investigate preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces.

Definition 6. A topological space X is said to be g-Hausdorff if whenever x and y are distinct points of X there are disjoint g-open sets U and V with $x \in U$ and $y \in V$.

It is obvious that every Hausdorff space is g-Hausdorff space.The following example shows that the converse is not true.

Example 1. Let $X = \{a, b, c\}$ and $\Im = \{\mathrm{f}, \{a\}, X\}$. It is clear that X is not Hausdorff. Since $\{a\}$, $\{b\}$ and $\{c\}$ are all g-open, it follows that X is g-Hausdorff.

Theorem 1. *Let X be a topological space and Y be Hausdorff. If $f : X \to Y$ is injective and g-continuous, then X is g-Hausdorff.*

Proof. Let x and y be any two distinct points of X Then $f(x)$ and $f(y)$ are distinct points of Y, because f is injective. Since Y is Hausdorff, there are disjoint open sets U and V in Y containing $f(x)$ and $f(y)$ respectively. Since f is g-continuous and $U \cap V = \phi$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint g-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is g-Hausdorff.

Theorem 2. *Let X be a topological space and Y be g-Hausdorff. If $f: X \to Y$ is injective and gc-irresolute, then X is g-Hausdorff.*

Proof. Let x and y be any two distinct points of X. Then $f(x)$ and $f(y)$ are distinct points of Y, because f is injective. Since Y is g-Hausdorff, there are disjoint g-open sets U and V in Y containing $f(x)$ and $f(y)$ respectively. Since f is gc-irresolute and $U \cap V = \phi$, we have have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint g-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is g-Hausdorff.

Definition 8**.** A topological space X is said to be rg-Hausdorff if whenever x and y are distinct points of X there are disjoint rg-open sets U and V with $x \in U$ and $y \in V$.

It is obvious that every g-Hausdorff space is an rg-Hausdorff space. The following example shows that the converse is not true.

Example 2. Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{a, b\}, X\}$. Since $\{a\}$, $\{b\}$ and $\{c\}$ are all rg-open sets, it implies that X is rg-Hausdorff. Since $\{c\}$ and $\{b, c\}$ are not g-open in X, it follows that a and c can not be separated by any two disjoint g-open sets in X. Hence X is not g-Hausdorff.

Theorem 3. *Let X be a topological space and Y be Hausdorff. If $f: X \to Y$ is injective and rg-continuous, then X is rg-Hausdorff.*

Proof. Let x and y be any two distinct points of X. Then $f(x)$ and $f(y)$ are distinct points of Y, because f is injective. Since Y is Hausdorff, there are disjoint open sets U and V in Y containing $f(x)$ and $f(y)$ respectively. Since f is rg-continuous and $U \cap V = \phi$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint rg-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is rg-Hausdorff.

Theorem 4. *Let X be a topological space and Y be rg-Hausdorff. If $f: X \to Y$ is injective and rg-irresolute, then X is rg-Hausdorff.*

Proof. Let x and y be any two distinct points of X Then $f(x)$ and $f(y)$ are distinct points of Y, because f is injective. Since Y is rg-Hausdorff, there are disjoint rg-open sets U and V in Y containing $f(x)$ and $f(y)$ respectively. Since f is rg-irresolute and $U \cap V = \phi$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint rg-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is rg-Hausdorff.

Conclusion

1. From the definitions of g-Hausdorff and rg-Hausdorff spaces, we obtain the following diagram:

X is a Hausdorff space $\Rightarrow$ X is a g-Hausdorff space $\Rightarrow$ X is a rg-Hausdorff space

2. We obtain preservation theorems on some topological spaces under g-continuous, rg-continuous, gc-irresolute and rg-irresolute functions. The results are the following:
   - 2.1 Let X be a topological space and Y be Hausdorff. If $f: X \to Y$ is injective and g-continuous, then X is g-Hausdorff.
   - 2.2 Let X be a topological space and Y be g-Hausdorff. If $f: X \to Y$ is injective and gc-irresolute, then X is g-Hausdorff.
   - 2.3 Let X be a topological space and Y be Hausdorff. If $f: X \to Y$ is injective and rg-continuous, then X is rg-Hausdorff.
   - 2.4 Let X be a topological space and Y be rg-Hausdorff. If $f: X \to Y$ is injective and rg-irresolute, then X is rg-Hausdorff.

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