Analytical solution of a 3D model for the airflow in a human oral cavity

Chatvarin Tasawang* and Supachara Kongnuan

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Khlong Luang, Pathum Thani, 12121

* Corresponding author. E-mail address: chatvarin.ts@gmail.com

Abstract

Understanding characteristic of the airflow in a human respiratory tract is very important factor to treatment in the respiratory disease. In this paper, we propose a three-dimensional mathematical modelling for the airflow in a human oral cavity. The airflow is assumed to be axially symmetric flow and driven by the oscillating pressure gradient. The governing equations for describing the behavior of airflow are composed the Navier-Stokes equations and the continuity equation in a cylindrical coordinates system. To solve the model, we presented method of analytical solution for the airflow velocity. We obtained a solution in a Fourier-Bessel series form. Then, we simulated the airflow field on a three-dimensional geometry of the oral cavity area. The obtained results show that the characteristic of magnitude and direction of the airflow correspond to the fact of the airflow in human airway and the previous research works.

Keywords: three-dimensional mathematical model, oral cavity, human upper respiratory tract, Navier-Stokes equations, analytical solution,

Introduction

In medical research, the aerosolized medicine is used to being the highly effective treatment for the respiratory disease because droplet particles are directly transported into the respiratory position. The droplet particles have also slight side effects. However, the success of the aerosolized treatment depends on the airflow behavior which is an important factor that transports droplet particles in respiratory position. The airflow is also determined trajectory and final location of the particles. Therefore, the characteristic of airflow and particle trajectory are studied extensively both in medical and mathematics. But the medical studies must take a lot of time and high-performance computers in the laboratory experiments, most of researchers use the mathematical model to describe the problem instead.

Since the air is one of fluids, a lot of researchers tend to express the airflow velocity by the Navier-Stokes equations. There are many works that presented a numerical method for finding solution of these equations and simulate the airflow field especially in a human respiratory tract, such as the research of Wang, Denney, Morrison, and Vodyanoy (2005) presented a numerical simulation of airflow in the human nasal cavity with computational fluid dynamics software and Qingxing, Fong, and Chi-Hwa (2009) studied the numerical solution of the Navier-Stokes equations for airflow and simulated particle trajectories in a human airway.

However, the numerical analysis is quite complex as well as using a high-performance computers, some researchers tried to solve of the Navier-Stokes equations by an analytical method for saving time and computing resources. We can see from some works, for instance, Tsangaris and Vlachakis (2003) presented an analytical solution for the fully developed laminar flow in duct of a cross-section of right-angled isosceles triangle. They have obtained solution in a Fourier series form. Otarod and Otarod (2006) studied an analytical solution for the two-dimensional laminar incompressible flow. Mohyuddin Siddiqui, Hayat, Siddqui, & Asghar (2008) presented the solutions of the Navier-Stokes equations governing the unsteady incompressible flow. The solutions have been obtained using Hodograph-Legendre transform method. Emin and Erdem (2009) showed an analytical solution of the Navier-Stokes equations governing for flow over a moving plate bounded by two side walls. Furthermore, Kongnuan and Pholuang (2012) proposed an analytical solution of the Navier-Stokes equations for the oscillating airflow in a human upper airway. Nevertheless, the analytical simulation for the three-dimensional model of the oscillating airflow in the human upper respiratory tract has never been presented.

Due to we interest the problem of the oscillating airflow in a human upper airway in the research of Kongnuan and Pholuang (2012), we then to develop the two-dimensional human upper airway model of these research to the three-dimensional model for a more realistic geometry. For this work, we will show the three-dimensional model of a human upper respiratory tract including oral cavity until the end of trachea.

We propose the mathematical model that the governing equations are described by the Navier-Stokes equations and the continuity equation in cylindrical coordinates for axially symmetric case with suitable boundary conditions. Then we present the method of analytical solution for the airflow velocity. The airflow field is simulated in the oral cavity area.

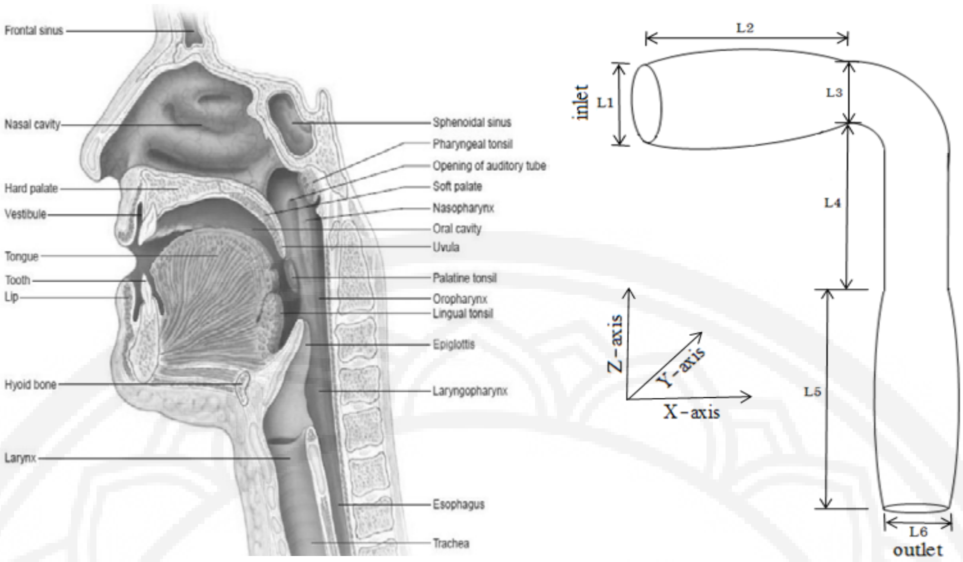
Methodology and Solution

Construction of Model

Because of the complex geometry of a human upper airway as shown in Figure 1(a), which is barrier to create a realistic domain and difficult to derive an analytical expression for the solution of the airflow, therefore, we look a three-dimensional airway in simple model that while the oral is opening wide, the inside of oral cavity seem like an ellipsoid tube shape and connect to trachea tube straight down. To simplify the simulation the airflow field on a 3D-model, we consider the oscillating airflow in the human upper airway for the axially symmetric flow case. Here, the domain of our model is started from the beginning of the oral cavity to the end of the trachea in a human upper airway as shown in Figure 1(b). The 3D-model generated from the parameters in Table 1 (Kongnuan & Pholuang, 2012).

Table 1 Parameters of a human upper respiratory tract.

Parameter	Length (cm)
The diameter of inlet (L1)	3.125
The length of oral cavity (L2)	7.0
The diameter of upper trachea (L3)	2.5
The length of upper trachea (L4)	7.0
The length of lower trachea (L5)	9.0
The diameter of outlet (L6)	2.0



(a) (b)

Figure 1 (a) The construction of a real human upper airway (Image source: Shier et al., Hole's Essentials of Human Anatomy and Physiology) and (b) 3D- human upper respiratory model.

For convenience to derive the analytical expression for the solution of the airflow velocity, we divide the domain of a upper airway into 4 areas as shown in Figure 2.

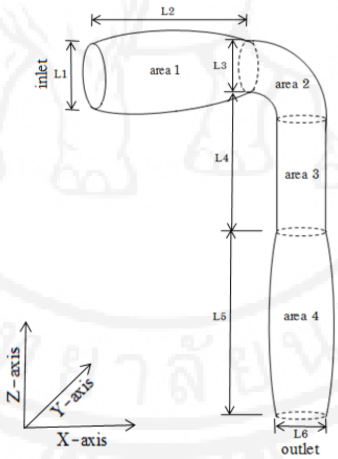


Figure 2 Division of the domain

Governing equations and boundary conditions

In this paper, we simulate the problem of the airflow only the oral cavity area under the assumptions that the air is an incompressible Newtonian fluid which has constant viscosity and density. The airflow is assumed that there is no effect from any external force and it is driven by the oscillating pressure gradient within the pulmonary. Therefore, we can use the Navier -Stokes equations and the continuity equation to be the governing equations to describe the airflow in airway.

Hence, the governing equations for the three-dimensional airflow model are composed the Navier-Stokes equations and the continuity equation in cylindrical coordinates system. These equations are given in terms of velocity components u_r, u_ϕ, u_x in cylindrical coordinates

$\mathbf{u} = u_r e_r + u_\phi e_\phi + u_x e_x,$

where e_r, e_ϕ, e_x are standard unit vectors in radial, angular, and axial direction, respectively.

By assumption that the flow is axially symmetric, we search for the axially symmetric solution, the angular velocity component u_ϕ set to zero. Then, the airflow velocity depends on only the radial direction r and the axial direction x. Therefore, the governing equations can be now written as follows form:

$$\frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_x}{\partial x} = 0 \quad , \tag{1}$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_x \frac{\partial u_r}{\partial x}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial x^2}\right), \tag{2}$$

$$\rho\left(\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + u_x \frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} + \frac{\partial^2 u_x}{\partial x^2}\right), \tag{3}$$

where u_x is the axial component of the flow velocity, u_r is the radial component of the flow velocity, p is the pressure, ρ and μ are the density and dynamic viscosity of the air, respectively.

The boundary conditions are described as follows: The non-slip boundary condition, $\mathbf{u} = (u_x, u_r) = (0,0)$, is assigned to the inner walls. The pressure at the inlet is zero, while the pressure at outlet is an oscillating function sine of time, $p(t) = P\sin(\omega t)$, (Kongnuan & Pholuang, 2012) where P is the amplitude of the oscillating pressure (Hranitz, 2009).

From equations (1)-(3) combine with the above boundary conditions, we now have a boundary value problem (BVP). Then we will find a solution of the BVP in the following section.

The method of analytical solution

To find the analytical solution in cylindrical coordinates for the oscillating airflow conditions in area 1 (Figure 2), we firstly transform model of area 1 which is in rectangular coordinates system (x, y, z) into cylindrical coordinates system (x, r) as the following:

Since the model in area 1 is look like the elliptic tube, we can consider the ellipsoid equation that is

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 + \left(\frac{z'}{c}\right)^2 = 1; \quad x' = x - x_1, \quad y' = y - y_1, \quad z' = z - z_1.$$

This equation is transformed to cylindrical coordinates system by defining $x' = x', y' = r\cos\varphi, z' = r\sin\varphi \quad \varphi = \tan^{-1}(\frac{z'}{y'})$, and we let $c = b$, we then get this equation as the following new

form: $\left(\frac{x'}{a}\right)^2 + \left(\frac{r}{b}\right)^2 = 1; \quad r^2 = (y')^2 + (z')^2.$

Now we obtain the model of area 1 in cylindrical coordinates as shown in Figure 3, which is confined by

$x=0,\quad x=a=L_2,\quad r=r_{upper}(x)=b_1\sqrt{(1-(\frac{x'}{a_1})^2)}+(L_5+L_4),\quad r=r_{lower}(x)=-b_1\sqrt{(1-(\frac{x'}{a_1})^2)}+(L_5+L_4)$ and $b=r_{upper}(x)-r_{lower}(x)$, where $a_1=x_1=\sqrt{(\frac{L_2}{2})^2/1-(\frac{L_1}{4})^2}$, and $b_1=L_5-L_4$.

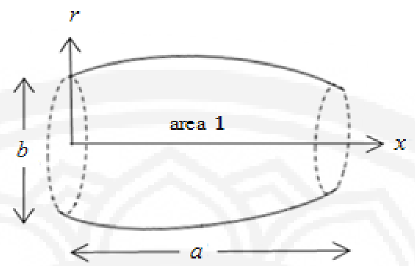


Figure 3 The oral cavity model in cylindrical coordinates system

Next, we use the analytical expression which is based on the Fourier-Bessel series form for the airflow in the oral cavity as shown in Figure 3. From equations (1)-(3), we see that these equations are three partial differential equations with three unknowns u_x, u_r and p as functions of three independent variables x, r and t. By assuming fully developed flow on area 1, so that $u_r=0$ and $u_x=u_x(t,r)$. The continuity equation is satisfied and when developed flow conditions across the $x-$axis, equation (2) is omitted. We are interested in the cases of the airflow which is driven by the oscillating pressure gradient, then the following partial differential equations satisfied:

$$\frac{\partial u_x}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u_x}{\partial r^2}+\frac{1}{r}\frac{\partial u_x}{\partial r}\right),\qquad \frac{\partial p}{\partial x}=\frac{P}{a}\sin(\omega t) \tag{4}$$

where $\frac{P}{a}$ is the amplitude of the pressure gradient, a is the length on $x-$axis of the considered region and ω is the cyclic frequency of the oscillating pressure gradient. The solution u_x is defined to be periodic function as the follow:

$$u_x(r,t)=u_s(r)\sin(\omega t)+u_c(r)\cos(\omega t). \tag{5}$$

In order to the accurately solution, dimensionless variables $\tilde{x}, \tilde{r}, \tilde{u}_x$ and α are introduced as

$$\tilde{r}=\frac{r}{b},\quad \tilde{x}=\frac{x'}{a},\quad \tilde{u}_x=\frac{u_x}{Pb^2}\mu a,\quad \alpha=b\sqrt{\frac{\omega\rho}{\mu}}, \tag{6}$$

where $b=b(x)=r_{upper}(x)-r_{lower}(x)$, is the length of $r-$axis of the considered region and α is the reduced frequency. After the introducing dimensionless variables, each region is transformed to be a one-unit region.

Equation (4) together with equation (5) are reduced to a system of non-homogeneous Helmholtz equations in one dimension (Rosu & Romero, 1999):

$$\alpha^2\tilde{u}_s=\frac{d^2\tilde{u}_c}{d\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{d\tilde{u}_c}{d\tilde{r}},\qquad -\alpha^2\tilde{u}_c=-1+\frac{d^2\tilde{u}_s}{d\tilde{r}^2}+\frac{1}{\tilde{r}}\frac{d\tilde{u}_s}{d\tilde{r}}. \tag{7}$$

The boundary conditions for $\tilde{u}_s$ and $\tilde{u}_c$ are stated as follows:

$$\tilde{u}_s(0)=0, \qquad \tilde{u}_c(0)=0, \qquad \tilde{u}_s(1)=0, \qquad \tilde{u}_c(1)=0. \tag{8}$$

The analytical solution of equation (7) which satisfies the boundary conditions (8) can be determined by using a Fourier-Bessel analysis of $\tilde{u}_s$, $\tilde{u}_c$ for $\tilde{r}$. Hence, $\tilde{u}_s$, $\tilde{u}_c$ and 1 are expressed (Gockenbach, 2011) as Fourier expansions:

$$\tilde{u}_s = \sum_{m=1}^{\infty} A_m J_0(\alpha_m \tilde{r}), \tag{9}$$

$$\tilde{u}_c = \sum_{m=1}^{\infty} B_m J_0(\alpha_m \tilde{r}), \tag{10}$$

$$1 = \sum_{m=1}^{\infty} C_m J_0(\alpha_m \tilde{r}), \tag{11}$$

where J_0 is the Bessel function of order zero which have an infinite number of positive roots α_m. The term J_0 given by

$$J_0(\alpha_m \tilde{r}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{\alpha_m \tilde{r}}{2}\right)^{2n}.$$

Substituting the above expansions (9)-(11) in the system of differential equations (7), we get the unknown coefficients A_m and B_m as follows:

$$A_m = \left(-\frac{1}{\alpha^4}\right)\left(\frac{(2n)(\alpha_m)^3(\alpha_m(2n-1)+1)}{4n^2-[(\alpha_m)^3(\alpha_m(2n-1)+1)]^2}\right)C_m, \tag{12}$$

$$B_m = \left(\frac{1}{\alpha^2}\right)\left(\frac{4n}{4n^2-[(\alpha_m)^3(\alpha_m(2n-1)+1]^2}\right)C_m; \qquad n=0,1,2,\ldots, \; m=1,2,3,\ldots$$

We can calculate coefficient C_m by the usual formula:

$$C_m = \frac{\int_0^1 J_0(\alpha_m \tilde{r})\tilde{r}d\tilde{r}}{\int_0^1 [J_0(\alpha_m \tilde{r})]^2 \tilde{r}d\tilde{r}}; \qquad m=1,2,3,\ldots \tag{13}$$

The resulting periodic velocity can be written as:

$$\tilde{u}_x = \tilde{u}_a \sin(\omega t), \qquad \tilde{u}_a = \sqrt{\tilde{u}_s + \tilde{u}_c}, \tag{14}$$

where $\tilde{u}_a$ is the amplitude resulting from the expression of $\tilde{u}_x$. We then can obtain the airflow velocity u_x by substituting $\tilde{u}_x$ back into equation (6) , we get

$$u_x = \frac{Pb^2}{\mu a}\tilde{u}_x. \tag{15}$$

Finally, we have obtained the analytical solution in cylindrical coordinates. We can transform back into the rectangular coordinates system as follows: $u_x = u_x$, $u_y = u_r \cos\varphi$, and $u_z = u_r \sin\varphi$, the components of the flow velocity in $x-$axis, $y-$axis, and $z-$axis, respectively. We then use the obtained solution to simulate the airflow field in the three-dimensional oral cavity which is the one part of a human upper airway.

Results Discussion

In this paper, the analysis is carried out with $\rho = 1.148\times10^{-3} g/m^3$, $\mu = 1.82\times10^{-5} Pa.s$ and amplitude of the oscillating intrapulmonary pressure

$P=-133.32Pa$. from the research in medical literatures of Zhang and Kleinstreuer (2004), they found that the period of human breathing is about 4s. Therefore, we use $\omega=\frac{\pi}{2}$.

In order to show the realistic simulation, we present the 3D arrow plot for airflow field. In Figures 4-5, we demonstrate the 3D arrow plots of airflow field in the breathing period at t = 1.4s and t = 2.5s, respectively. In addition, to see more clearly the airflow field, the arrow plot of airflow field is also plotted in the XZ-plane for the central cavity $(y'=0)$, left side $(y'=1)$ and right side $(y'=-1)$ of the oral cavity at t = 1.4s and t = 2.5s as shown in Figures 6-7. For the present magnitude of the velocity, the contour plot of the airflow velocity for the central cavity $(y'=0)$ of the oral cavity at = 1.4s and t = 2.05s are shown in Figure 8.

When we consider the 3D arrow plot of the airflow velocity as shown in Figures 4-5, it is found that the air flows into the oral cavity at t = 1.4s which corresponds to the pulmonary relaxation. In contrast, the air flows out the oral cavity at t = 2.5s which corresponds to pulmonary contraction. At the same locations, the velocity of airflow has size and direction different by each time. The maximum velocity occurs in the central area of the oral cavity and reduces to the zero value close to the walls.

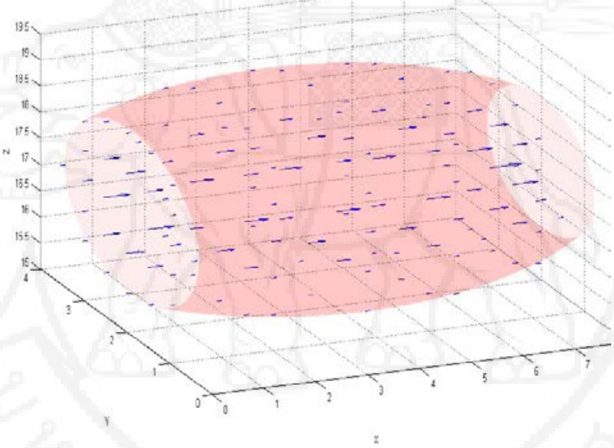


Figure 4 The 3D arrow plot of airflow field in the oral cavity at t = 1.4 s

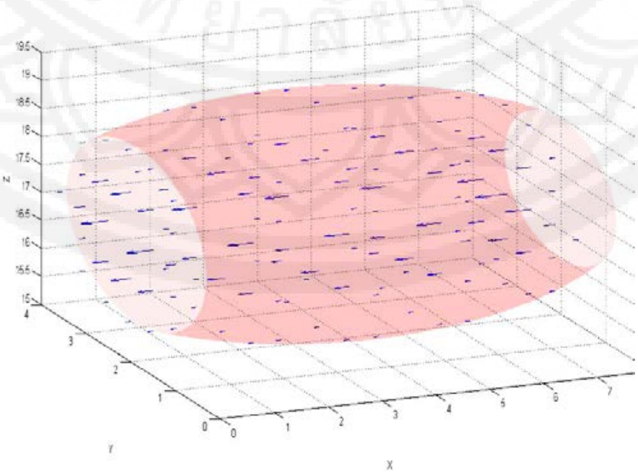


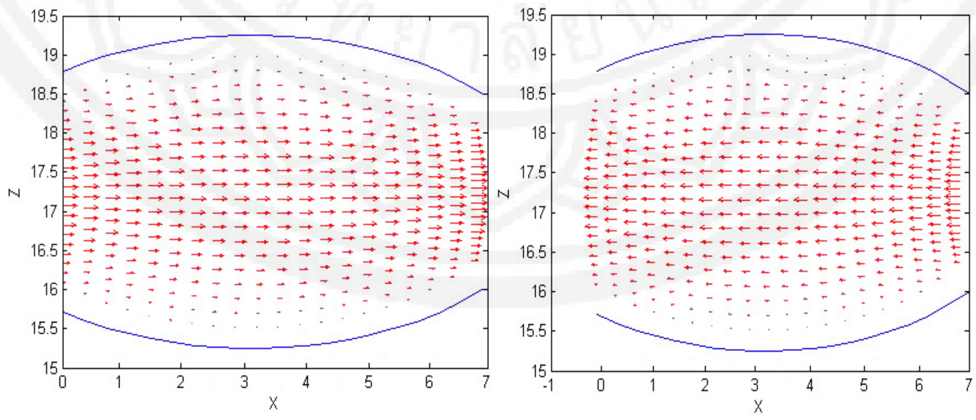
Figure 5 The 3D arrow plot of airflow field in the oral cavity at t = 2.5 s

In addition, the arrow plot of airflow field in the XZ-plane at the central cavity as shown in Figures 6(a)-6(b), we see clearly the flow that the velocity profiles become flat in the central area and reduce to the zero for the area which is more close to the walls. When air flows into the oral cavity at t = 1.4s, the velocity shows the maximum value that about 549 cm/s in the central area and when air flows out the oral cavity at t = 2.5s, the maximum velocity value have about 480 cm/s.

At the sides of cavity as shown in Figures 7(a)-(d), the cross-sectional sides of oral cavity are less than central oral cavity. We found that when t = 1.4s air flows into the oral cavity and the velocity have maximum value that about 273 cm/s in the central of the plane. In contrast, when t = 2.5s air flows out the oral cavity and the velocity have maximum value that about 238 cm/s in the central of the plane. We can see that the central oral cavity have maximum value of the velocity more than at the side of oral cavity.

When we consider the contour plot of magnitude of the airflow velocity in the central cavity ($y' = 0$) at t = 1.4s and t = 2.05s as shown in Figures 8(a)-8(b), it is found that the velocity magnitude become flat in the central of the oral cavity. This shows a boundary layer behavior with a high velocity gradient close to the boundaries. The magnitude of velocity always shows the maximum value in the central area and reduces to the zero value for the area which is more close to the walls, which correspond to the arrow plot. When compare the velocities at different times are different and when t = 2.05s, the direction of the flow is changing to become the out flow, their magnitudes of the velocity are less than other times and close to zero.

From the previous results, we can conclude that (1) the airflow has changed both magnitude and direction according to the breathing period. By during the first two seconds, the air flows into the oral cavity and the direction of the flow is changing to become the out flow at the beginning of the third seconds. (2) At the same locations, the magnitude and direction of airflow velocity are changed and depend on time. The maximum velocity occurs in the central area of the oral cavity and reduces to the zero value to the wall. The obtained velocities are in the range of [0,900] cm/s which agree to other publications (Kongnuan & Pholuang, 2012; Zhang & Kleinstreuer, 2004).



(a) (b)

Figure 6 The XZ-plane arrow plot of airflow field in central of the oral cavity ($y' = 0$) at (a) t = 1.4 s and (b) t = 2.5 s

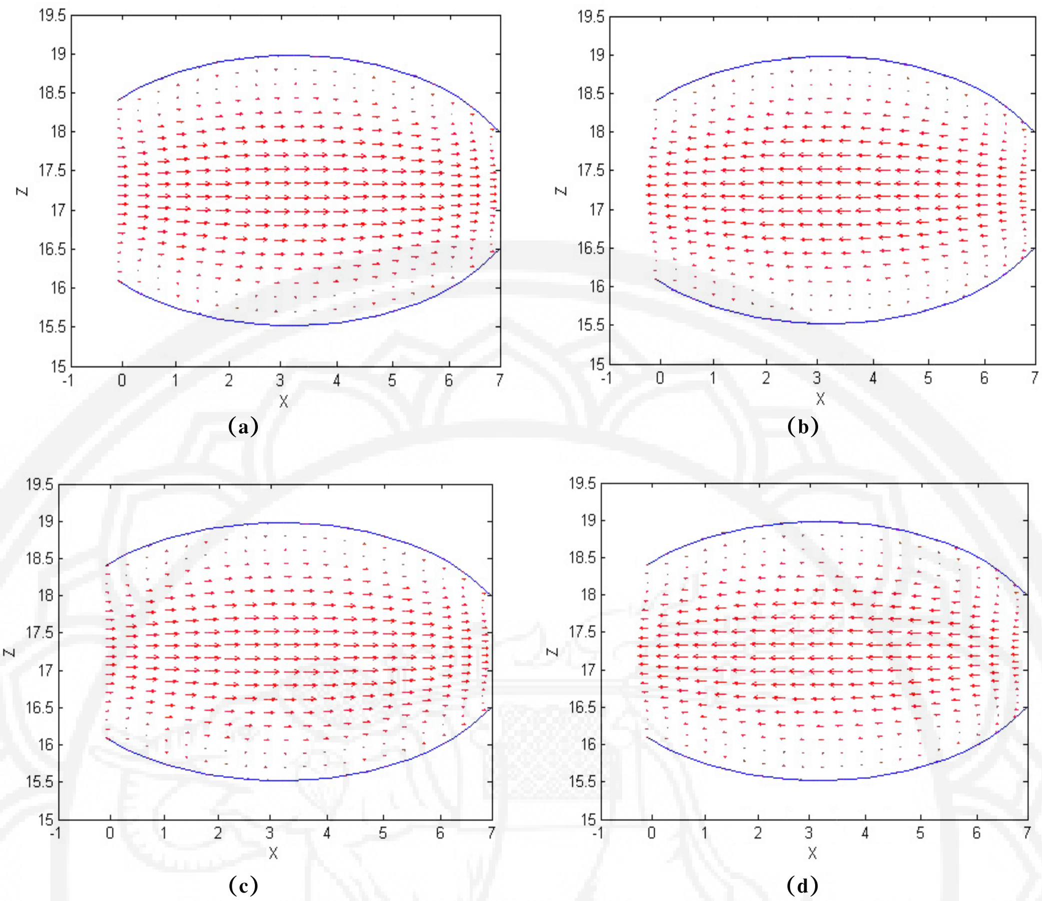


Figure 7 The XZ-plane arrow plot of airflow field in the side of oral cavity at t = 1.4 s (left) and t = 2.5 s (right) for: (a) and (b) right side of the oral cavity $(y' = -1)$, (c) and (d) left side of the oral cavity $(y' = 1)$.

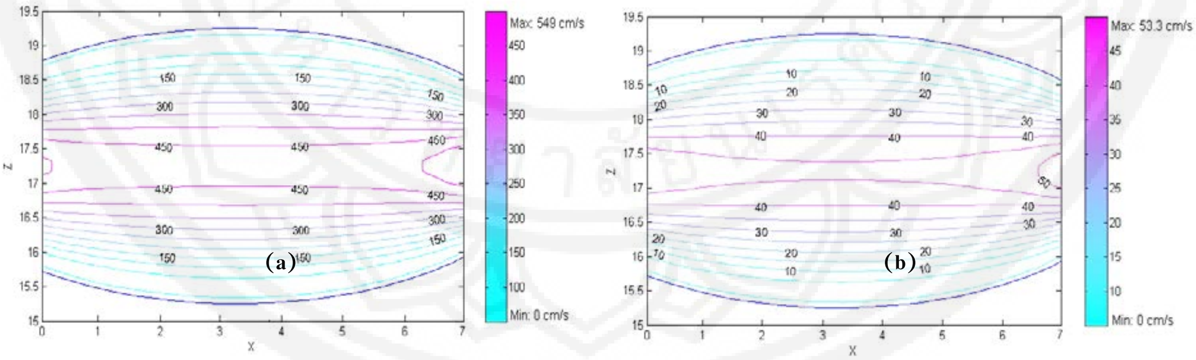


Figure 8 The XZ-plane contour plot of magnitude of the airflow velocity in central of the oral cavity $(y' = 0)$ at (a) t = 1.4 s and (b) t = 2.05 s.

Conclusions

An analytical expression for the axially symmetric solution of the airflow in human oral cavity model which is described by the Navier-Stokes and continuity equations in cylindrical coordinates are carried out as the objective of this research. The analytical solution of the oscillating airflow is given in a Bessel series form. The three-dimensional mathematical model is sophisticated to simulate the airflow velocity. The obtained airflow profiles are reasonable and correspond to the previous works (Kongnuan & Pholuang, 2012; Zhang & Kleinstreuer, 2004).

Acknowledgements

The authors would like to thank Department of Mathematics and Statistics, Thammasat University for giving the facilities support this work.

References

Emin, M., & Erdem, C. (2009). An analytical Solution of the Navier-Stokes Equation for Flow over a Moving Plate Bounded by Two Side Walls. *Journal of Mechanical Engineering*, *55*(12), 749-754.

Gockenbach, M. S. (2011). *Partial differential equations: analytical and numerical methods* (2nd ed.). The United States of America: The Society for Industrial and Applied Mathematics.

Hranitz, M. J. (2009). *APHNT: Respiratory Physiology Outlines*. Retrieved from http://facstaff.bloomu.edu/jhranitz/Courses/APHNT/Outlines/Respir

Kongnuan, S., & Pholuang, J. (2012). A Fourier Series-Based Analytical Solution for the oscillating Airflow in a Human Respiratory Tract. *International journal of pure and applied mathematics*, *78*(5), 721-733.

Mohyuddin, M. R., Siddiqui, A. M., Hayat, T., Siddqui, J., & Asghar, S. (2008). Exact solutions of time-dependent Navier-Stokes equations by Hodograph-Legendre transformation method. *Tamsui Oxford Journal of Mathematical Sciences*, *24*(3), 257-268.

Qingxing, X., Fong, Y. L., Chi-Hwa, W. (2009). Transport and deposition of inertial aerosols in bifurcated tubes under oscillatory flow. *Chemical Engineering Science*, *64*, 830-846.

Otarod, O., & Otarod, D. (2006). *Analytical solution for Navier-Stokes equations in two dimensions for laminar incompressible flow*. Retrieved from http://arxiv.org/physics/0609186

Rosu, H., & Romero, J. L. (1999). Ermakov approach for the one-dimensional Helmholtz Hamiltonian. *Nuovo Cimento*, *114*, 569-574.

Tsangaris, S., & Vlachakis, N. W. (2003). Exact solution of the Navier-Stokes equations for the oscillating flow in duct of a cross-section of right-angled isosceles triangle. *Z. angew. Math. Phys.*, *54*, 1094-1100.

Wang, K., Denney, T. S., Morrison, E. E., & Vodyanoy, V. J. (2005). *Numerical simulation of air flow in the human nasal cavity*. Retrieved from http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=1615757

Zhang, Z., & Kleinstreuer, C. (2004). Airflow structures and nano-particle deposition in a human upper airway model. *Journal of Computational Physics*, *198*, 178-210.