Left Regular and Right Regular Elements of the Semigroups of Transformations Restricted by an Equivalence

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Abstract

Let $T(X)$ be the semigroup of all transformations on a set X. For an arbitrary equivalence relation σ on X, we consider a subset of $T(X)$ defined by

$$E(X,\sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \Longrightarrow x\alpha = y\alpha\}.$$

It is obvious that $E(X,\sigma)$ is a subsemigroup of $T(X)$. In this paper, we characterize the left regular, the right regular and the completely regular for elements of $E(X,\sigma)$. Moreover, we give a necessary and sufficient conditions for the semigroup $E(X,\sigma)$ when it is left regular, right regular and completely regular, respectively.

Keywords: transformation semigroup, equivalence relation, left regular, right regular, completely regular

Introduction

An element a of a semigroup S is called *left regular* if $a = xa^2$ for some $x \in S$, *right regular* if $a = a^2x$ for some $x \in S$ and *completely regular* if $a = axa$ and $ax = xa$ for some $x \in S$. In fact, every completely regular element is left regular and right regular. Moreover, Petrich and Reilly (1999) proved that an element a of a semigroup S is completely regular if and only if a is both a left and a right regular element of S. If all its elements of S are left (right, completely) regular, we call S a *left* (*right*, *completely*) *regular semigroup*.

The full transformation semigroup on a non-empty set X is denoted by $T(X)$, that is, $T(X)$ is the semigroup of all mappings $\alpha : X \to X$ under the composition. Particularly, characterization of regularity on subsemigroups of $T(X)$ have been investigated, see Choomanee, Honyam and Sanwong (2013); Laysirikul (2016); Laysirikul and Namnak (2013); Namnak and Laysirikul (2013); and Sirasuntorn and Kemprasit (2010). In Pei (2005), the author studied a subsemigroup of $T(X)$ determined by an arbitrary equivalence relation σ, namely

$$T(X,\sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \Longrightarrow (x\alpha, y\alpha) \in \sigma\}.$$

He investigated regularity and Green's relations for $T(X,\sigma)$. After, Pei and Deng (2009) described the equivalence relation σ on X for which Green's relations $\mathcal{D}$ and $\mathcal{J}$ are coincided in the semigroup $T(X,\sigma)$. In 2013, Namnak and Laysilikul (2013) investigated a necessary and sufficient conditions when elements of $T(X,\sigma)$ to be left regular, right regular and completely regular. Recently, Mendes-Gonçalves and Sullivan (2010) introduced a subsemigroup of $T(X)$ defined by

$$E(X,\sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \Longrightarrow x\alpha = y\alpha\}$$

and call it the *semigroup of transformations restricted by an equivalence* σ. Then $E(X,\sigma)$ is a subsemigroup of $T(X,\sigma)$. If $\sigma = I_X$ where I_X is the identity relation on X, then $E(X,\sigma) = T(X,\sigma)$. The authors characterized Green's relations on the largest regular subsemigroup of $E(X,\sigma)$. They also showed that if $|X| \geq 2$ and $\sigma \neq I_X$, then $E(X,\sigma)$ is not isomorphic to $T(Z)$ for any set Z.

The aim of this paper is to characterize left regular, right regular and completely regular elements of $E(X,\sigma)$, respectively. We also investigate a condition for which of the semigroup to be left regular, right regular and completely regular.

In what follows, let σ be an equivalence relation on a nonempty set X and the quotient set is denoted by X/σ.

Main Results

We first introduce the following terminology. For $\alpha \in T(X)$, the symbol $\pi(\alpha)$ will denote the decomposition of X induced by the map α, namely

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}.$$

Hence $\pi(\alpha) = X/\ker\alpha$ where $\ker\alpha = \{(x,y) \in X \times X : x\alpha = y\alpha\}$.

The following lemma is needed for characterization of left regular.

Lemma 1. Let $\alpha \in E(X,\sigma)$. For each $A \in X/\sigma$, there exists $P \in \pi(\alpha)$ such that $A \subseteq P$.

Proof. Let $A \in X/\sigma$ and $a \in A$. Choose $P = (a\alpha)\alpha^{-1}$. If $x \in A$, then $x\alpha = a\alpha$ and hence $x \in (a\alpha)\alpha^{-1} = P$. Therefore $A \subseteq P$.

Now, we investigate the condition under which an element in $E(X,\sigma)$ is left regular.

Theorem 2. Let $\alpha \in E(X,\sigma)$. Then α is left regular if and only if for every $P \in \pi(\alpha)$, there exists $x \in X$ such that $x\alpha \in P$.

Proof. Assume that α is left regular. Then $\alpha = \beta\alpha^2$ for some $\beta \in E(X,\sigma)$. Let $P \in \pi(\alpha)$ and $y \in P$. Then

$$y\alpha = y\beta\alpha^2 = y\beta\alpha\alpha$$

and hence $y\beta\alpha \in (y\alpha)\alpha^{-1} = P$. Therefore for any $P \in \pi(\alpha)$, there is $x = y\beta \in X$ such that $x\alpha \in P$.

Conversely, for each $P \in \pi(\alpha)$, we choose and fix an element $x_P \in X$ such that $x_P\alpha \in P$. Let $x \in X$. Since $\pi(\alpha)$ is a partition of X, there exists $P_x \in \pi(\alpha)$ such that $x \in P_x$. We then have $x_{P_x}\alpha\alpha = x\alpha$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

Let $x, y \in X$ be such that $(x,y) \in \sigma$. Then there exists $A \in X/\sigma$ such that $x, y \in A$. By Lemma 1, there is $P \in \pi(\alpha)$ such that $A \subseteq P$. Thus $x, y \in P$ and so $x_{P_x} = x_P = x_{P_y}$. This implies that $x\beta = y\beta$. Hence $\beta \in E(X,\sigma)$. If $x \in X$, then $x\beta\alpha^2 = x_{P_x}\alpha\alpha = x\alpha$ which gives $\alpha = \beta\alpha^2$. We conclude that α is left regular.

An element x of a semigroup S is called *idempotent* if $x^2 = x$. Clearly, if x is idempotent, then x is both a left and a right regular element. For $\alpha \in T(X)$, we have that every constant mapping is an idempotent element. If $\sigma = X \times X$, then every element of $E(X,\sigma)$ is constant. Hence every element of $E(X,\sigma)$ is idempotent.

Next, using the fact above proves a necessary and sufficient conditions for the semigroup $E(X,\sigma)$ which is left regular.

Theorem 3. If $|X| \leq 2$, then $E(X,\sigma)$ is a left regular semigroup.

Proof. Suppose that $|X| \leq 2$. If $|X| = 1$, then $E(X,\sigma)$ contains only one element. Clearly, $E(X,\sigma)$ is a left regular semigroup. Assume that $|X| = 2$. Then $\sigma \in \{I_X, X \times X\}$. If $\sigma = X \times X$, then every element of $E(X,\sigma)$ is idempotent and hence $E(X,\sigma)$ is a left regular semigroup. If $\sigma = I_X$, then we have

$$E(X,\sigma) = \left\{\begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix}\right\}$$

where $X = \{a,b\}$. It is easy to see that $E(X,\sigma)$ is a left regular semigroup by Theorem 2.

Theorem 4. Let $|X| \geq 3$. Then $E(X,\sigma)$ is a left regular semigroup if and only if $\sigma = X \times X$.

Proof. Suppose that $\sigma \neq X \times X$. Then there exist distinct elements $A, B \in X/\sigma$. Let $a \in A$ and $b \in B$. Since $|X| \geq 3$, there is an element $c \in X \setminus \{a,b\}$. We distinguish two cases.

Case 1. Either $c \in A$ or $c \in B$. Without loss of generality, we may assume that $c \in A$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ c & \text{otherwise.} \end{cases}$$

Then $\alpha \in E(X,\sigma)$. Let $P = c\alpha^{-1}$. Then $P \in \pi(\alpha)$ and $P = X \setminus A$. Since $a, c \in A$, $P \cap X\alpha = \emptyset$. Therefore $x\alpha \notin P$ for all $x \in X$ and thus α does not satisfy Theorem 2. Hence α is not a left regular element of $E(X,\sigma)$.

Case 2. $c \notin A$ and $c \notin B$. Then there is $C \in X/\sigma$ such that $c \in C$. Thus $C \notin \{A,B\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A \cup B, \\ b & \text{otherwise.} \end{cases}$$

Obviously, $\alpha \in E(X,\sigma)$ and the set $P = b\alpha^{-1} \in \pi(\alpha)$. Then $P = X \setminus (A \cup B)$ and so $P \cap X\alpha = \emptyset$. Hence by Theorem 2 we obtain that α is not a left regular element of $E(X,\sigma)$.

From the two cases, we conclude that $E(X,\sigma)$ is not a left regular semigroup.

Conversely, if $\sigma = X \times X$, then every element of $E(X,\sigma)$ is idempotent and hence $E(X,\sigma)$ is a left regular semigroup.

Next, we give a characterization of the right regular elements in $E(X,\sigma)$.

Theorem 5. Let $\alpha \in E(X,\sigma)$. Then α is right regular if and only if $\alpha|_{X\alpha}$ is an injection.

Proof. Assume that α is right regular. Then $\alpha = \alpha^2\beta$ for some $\beta \in E(X,\sigma)$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Then $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Thus $x = x'\alpha = x'\alpha^2\beta = x\alpha\beta = y\alpha\beta = y'\alpha^2\beta = y'\alpha = y$. This means that $\alpha|_{X\alpha}$ is an injection.

Conversely, suppose that $\alpha|_{X\alpha}$ is an injection. Let $A \in X/\sigma$ be such that $A \cap X\alpha^2 \neq \emptyset$. We choose and fix an element $x_A \in A \cap X\alpha^2$. For each $x \in A \cap X\alpha^2$, there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$ by assumption. We observe that $(x'\alpha, x_A'\alpha) = (x, x_A) \in \sigma$. This implies that $x'\alpha\alpha = x_A'\alpha\alpha$. By assumption, we get that $x'\alpha = x_A'\alpha$. Since $x', x_A' \in X\alpha$ and $\alpha|_{X\alpha}$ is injective, $x' = x_A'$. Define $\beta_A : A \to X$ by

$$x\beta_A = x_A' \quad \text{for all } x \in A.$$

Then we define the map $\beta : X \to X$ by

$$\beta|_A = \begin{cases} \beta_A & \text{if } A \cap X\alpha^2 \neq \emptyset, \\ c_A & \text{otherwise,} \end{cases}$$

for all $A \in X/\sigma$ where c_A is the constant mapping from A into X. Since X/σ is a partition of X, β is well-defined. Obviously, $\beta \in E(X,\sigma)$. Finally, we will show that $\alpha = \alpha^2\beta$. Let $x \in X$, so $x\alpha^2 \in X\alpha^2$. Then there exists $A \in X/\sigma$ such that $x\alpha^2 \in A$. By the definition of β_A, $x\alpha^2\beta_A = (x\alpha^2)_A' = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$. By the uniqueness of $(x\alpha^2)'$, we obtain that $(x\alpha^2)' = x\alpha$. Thus $x\alpha^2\beta = x\alpha^2\beta_A = x\alpha$. We conclude that α is right regular, as asserted.

The proof of the next result is similar to Theorem 3.

Theorem 6. If $|X| \leq 2$, then $E(X,\sigma)$ is a right regular semigroup.

Theorem 7. Let $|X| \geq 3$. Then $E(X,\sigma)$ is a right regular semigroup if and only if $\sigma = X \times X$.

Proof. Assume that $\sigma \neq X \times X$. Then there are $A, B \in X/\sigma$ with $A \neq B$. Let $a \in A$ and $b \in B$. From $|X| \geq 3$, we let $c \in X \setminus \{a, b\}$.

Case 1. Either $c \in A$ or $c \in B$. Without loss of generality, we let $c \in A$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ c & \text{otherwise.} \end{cases}$$

Then $\alpha \in E(X,\sigma)$. Since $a, c \in X\alpha$ and $a\alpha = c\alpha$, $\alpha|_{X\alpha}$ is not injective.

Case 2. $c \notin A$ and $c \notin B$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A \cup B, \\ b & \text{otherwise.} \end{cases}$$

Obviously $\alpha \in E(X,\sigma)$. Since $c \notin A$ and $c \notin B$, $a, b \in X\alpha$. Note that $a\alpha = b\alpha$. Thus $\alpha|_{X\alpha}$ is not injective.

Form the discussion above, they follow from Theorem 5 that $E(X,\sigma)$ is not a right regular semigroup.

The converse of theorem is clear.

Finally, we give a characterization of the completely regular elements in $E(X,\sigma)$.

Theorem 8. Let $\alpha \in E(X,\sigma)$. Then α is completely regular if and only if $|P \cap X\alpha| = 1$ for all $P \in \pi(\alpha)$.

Proof. Suppose that α is a completely regular element. Then α is a left and a right regular element. Let $P \in \pi(\alpha)$. By Theorem 2, there exists $x \in X$ such that $x\alpha \in P$. Thus $P \cap X\alpha \neq \emptyset$. If $x, y \in P \cap X\alpha$, then $x\alpha = y\alpha$. It follows from Theorem 5 that $x = y$. Hence $|P \cap X\alpha| = 1$, as required.

Conversely, suppose that for each $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Then $x, y \in P \cap X\alpha$ for some $P \in \pi(\alpha)$. By assumption, we obtain that $x = y$, so that $\alpha|_{X\alpha}$ is an injection. By Theorem 5, we have α is right regular. From assumption and Theorem 2, we get that α is left regular. Hence α is completely regular.

Theorems 3, 4, 6 and 7 can be summarized as follows:

Corollary 9. $E(X,\sigma)$ is a completely regular semigroup if and only if $|X| \leq 2$ or $\sigma = X \times X$.

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