The Group Inverse of $3 \times 3$ Matrices Over a Ring.

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Abstract

In this paper, we study conditions for the existence of the group inverse of the $3 \times 3$ matrix

$$
N = \begin{bmatrix}
    e & h_1 & h_2 \\
    v_1 & a & c \\
    v_2 & b & d
\end{bmatrix}
$$

over an arbitrary ring $R$ with unity 1, when $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is the submatrix of $N$. has the group inverse in $R^{2 \times 2}$.

Keywords: von Neumann regularity, $\{1,2\}$-inverse, Group inverse, Matrix over a ring.

Introduction

Let $\mathbb{C}$ and $\mathbb{R}$ be the field of complex numbers and real numbers respectively. For a positive integers $m, n$, let $\mathbb{C}^{m \times n}$ be the set of all $m \times n$ matrices over $\mathbb{C}$. The set of all complex vectors, or $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^{n \times 1}$. We denote the identity and the zero matrix in $\mathbb{C}^{m \times n}$ by $I_m$ and $O_{m \times n}$, respectively. Note that $A^\dagger$ stands for $(A^*)^{-1}$.

For a given $A \in \mathbb{C}^{m \times n}$, the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying

1. $AX = A$
2. $XAX = X$
3. $(AX)^\dagger = AX$
4. $(XA)^\dagger = XA$

is called the Moore–Penrose inverse of $A$ and is denoted by $A^\dagger$ (see Ben–Israel, & Greville, 2003).

We also consider the following equations which are applicable to square matrices

$AX = X$ \hspace{1cm} (5)

For a given $A \in \mathbb{C}^{m \times m}$, the unique matrix $X \in \mathbb{C}^{m \times m}$ satisfying (1), (2), and (5) is called the group inverse of $A$ and denoted by $A^#$.

Unlike the Moore–Penrose inverse, which always exists, the group inverse need not exist for all square matrices. A well known necessary and sufficient condition for the existence of $A^#$ is that $\text{rank}(A) = \text{rank}(A^2)$. If $A$ is nonsingular, then $A^# = A^{-1} = A^\dagger$.

The group inverse has applications in singular differential and difference equations, Markov chains and iterative methods. Heinig, 1997, pp. 321–342 investigated the group inverse of Sylvester transformation. Wei, & Diao, 2005, pp. 109–123 studied the representation of the group inverse of a real singular Toeplitz matrix which arises in scientific computing and engineering. Catral, Olesky, &
Driessche, 2008, pp. 219–233 studied the existence of $A^\delta$ (see Cao, Ge, Wang, & Zhang, 2013).

If $\mathbf{A}$ and $\mathbf{B}$ are square invertible matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$. However, for a generalized inverse this need not be true. Rajesh Kannan & Bapat, 2014 (Theorem 2.2) asserted that for $\mathbf{A} \in \mathbb{C}^{m \times n}$ and $\mathbf{B} \in \mathbb{C}^{n \times p}$, $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$ if and only if $\mathbf{B} \mathbf{A}^\dagger$ and $\mathbf{A}^\dagger \mathbf{B}^\dagger$ are Hermitian.

Preliminaries and Auxiliary Results

We shall be now concerned with generalized inverses that satisfy some, but not all, of the four Penrose equations.

**Definition 2.1.** [Ben–Israel & Greville, 2003, p. 40].

For any $\mathbf{A} \in \mathbb{C}^{m \times n}$, let $\mathcal{A}(i,j,\ldots,k)$ denote the set of the matrices $\mathbf{X} \in \mathbb{C}^{m \times n}$ which satisfies equations $(i), (j), \ldots, (k)$ from among equations (1) - (4). A matrix $\mathbf{X} \in \mathcal{A}(i,j,\ldots,k)$ is called an $[i,j,\ldots,k]$-inverse of $\mathbf{A}$, and also denoted by $\mathbf{A}^{(i,j,\ldots,k)}$.

The examples are $[1]$-inverse (inner inverse), $[1,2]$-inverse (reflexive inner inverse), $[1,3]$-inverse (least squares inner inverse), $[1,4]$-inverse (minimum norm inner inverse), $[1,2,3]$-inverse, $[1,2,4]$-inverse and $[1,2,3,4]$-inverse, the last being the Moore–Penrose inverse of $\mathbf{A}$.

Let $\mathcal{M}$ be a ring with unity 1. An element $\mathbf{a} \in \mathcal{M}$ is said to be von Neumann regular (regular) if there exists an element $\mathbf{a}^* \in \mathcal{M}$ such that $\mathbf{aa} = \mathbf{a}$. In this case, $\mathbf{a}^*$ is called a [1]-inverse of $\mathbf{a}$. An element $\mathbf{a}^*$ of $\mathcal{M}$ is a $[1,2]$-inverse of $\mathbf{a}$, which is given by $\mathbf{a}^* = \mathbf{a} \mathbf{a}^\dagger \mathbf{a}$ (see [4] for example). Let $\mathcal{M}$ be a ring, not necessarily commutative.

Recall that an element $\mathbf{a} \in \mathcal{M}$ is said to be a unit if it has an inverse, if there is an element $\mathbf{a}^{-1} \in \mathcal{M}$ such that $\mathbf{a} \mathbf{a}^{-1} = \mathbf{a}^{-1} \mathbf{a} = 1$ (see for example Bhaskara Rao, 2002 (p.16). We denote an arbitrary $[1]$-inverse of $\mathbf{A}$ by $\mathbf{A}^+$ and $[1,2]$-inverse of $\mathbf{A}$, which is given by $\mathbf{A}^+ = \mathbf{A}^\dagger \mathbf{A}^\dagger$. In the next section we will use the following result on regularity.

**Lemma 2.2.** Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be regular, $\mathbf{B} \in \mathbb{R}^{n \times p}$, be such that there exists $\mathbf{A}'$ such that $\mathbf{A}' \mathbf{B} = \mathbf{B}^\dagger \mathbf{A} = \mathbf{O}_r$. If $(\mathbf{A} + \mathbf{B})$ is regular then $\mathbf{B}^\dagger$ is regular and

$(\mathbf{A} + \mathbf{B})^\dagger = (\mathbf{B}^\dagger)^{-1} + (\mathbf{B}^\dagger)^{-1} \mathbf{A}^\dagger$. is a $[1]$-inverse of $\mathbf{B}$, for any $(\mathbf{A} + \mathbf{B})^\dagger$.

**Proof:** Since $\mathbf{A}' = \mathbf{B}^\dagger \mathbf{A}$, then

$\mathbf{B} = (\mathbf{I}_m - \mathbf{A}' \mathbf{A}) (\mathbf{A} + \mathbf{B}) = (\mathbf{A} + \mathbf{B}) (\mathbf{I}_m - \mathbf{A}' \mathbf{A})$. Hence

$\mathbf{B} (\mathbf{A} + \mathbf{B})^\dagger = (\mathbf{I}_m - \mathbf{A}' \mathbf{A}) (\mathbf{A} + \mathbf{B}) (\mathbf{A} + \mathbf{B}) (\mathbf{I}_m - \mathbf{A}' \mathbf{A}) = (\mathbf{I}_m - \mathbf{A}' \mathbf{A}) (\mathbf{A} + \mathbf{B}) (\mathbf{I}_m - \mathbf{A}' \mathbf{A}) = \mathbf{B}$.

Therefore, $\mathbf{B}^\dagger$ is regular and $(\mathbf{A} + \mathbf{B})^\dagger$ is an $[1]$-inverse of $\mathbf{B}$.

Recently, Patricio, & Hartwig, 2010 characterized the existence of the group inverse of a two by two matrix with zero (2,2) entry, over an arbitrary ring. Castro–Gonzalez, Robles, & Velez–Cerrada, 2013 gave the conditions for the existence of the $2 \times 2$ the matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ has group inverse in $\mathbb{R}^{2 \times 2}$ in a matrix over rings $\mathcal{M}$, and derived a representation of the group inverse of $\mathcal{M}$ in the case when either the entry $a$ or $d$ has a group inverse in $\mathcal{M}$. Cao, et al., 2013 studied the group inverse of $2 \times 2$ block matrices $M = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ over rings $\mathcal{M}$ with unity 1, where

$\mathbf{CA} = \mathbf{CB} = \mathbf{0}$, and the group inverse of $\mathbf{D} - \mathbf{CB}$ exists. In this paper, we study the group inverse of $3 \times 3$ matrices

$M = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$

over an arbitrary ring $\mathcal{M}$ with unity 1, under the condition the $2 \times 2$ submatrix $\mathcal{M}$ has group inverse.

Now we assume that $\mathcal{M} \in \mathcal{R}^{2 \times 2}$ is regular and that $\mathcal{M}^\dagger$ is a fixed but arbitrary $[1,2]$-inverse of $\mathcal{M}$. Let us introduce the notation

$E = (\mathbf{I}_3 - \mathbf{MM}^\dagger)$, $F = (\mathbf{I}_3 - \mathbf{M}^\dagger\mathbf{M})$, $s = e - h^\dagger\mathbf{M}^\dagger v$. (2.2)

We note that $FF = F$ and $EE = E$, since
\[ EE = (I_2 - MM^+)(I_2 - MM^+) = (I_2 - MM^+) - MM^+ (I_2 - MM^+) = I_2 - MM^+ + MM^+ (I_2 - MM^+) = I_2 - MM^+ + MM^+ M^+ = I_2 - MM^+ = E, \]

and


We can decompose the matrix \( N \in R^{3 \times 3} \) as follows.

**Lemma 2.** The matrix \( N \) in (2.1) can be factored into

\[ N = \begin{bmatrix} 1 & h^T M^+ & s \end{bmatrix} \begin{bmatrix} s & h^T F & 1 \\ Ev & M & 0^T \\ 0 & I_2 & \end{bmatrix} \begin{bmatrix} Ev \\ M \\ I_2 \end{bmatrix} = XYZ^+ \quad (2.3) \]

where

\[ X = \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix}, \quad Z = \begin{bmatrix} s & h^T F \\ Ev & M \\ \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0^T \\ M^+ & I_2 \end{bmatrix} \]

**Proof:** Consider

\[ \begin{bmatrix} s & h^T F \\ Ev & M \\ \end{bmatrix} \begin{bmatrix} 1 & h^T M^+ \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} s + h^T F h^T M^+ & s h^T F + h^T M^+ \end{bmatrix} \begin{bmatrix} 1 \\ M^+ \end{bmatrix} = \begin{bmatrix} s + h^T F h^T M^+ & s h^T F + h^T M^+ \\ Ev & M \end{bmatrix} \begin{bmatrix} 1 \\ M^+ \end{bmatrix} \]

We have a useful characterization of \( N^+ \) as in [4, Lemma 1.2].

**Proof:** We must show that,

\[ EM = MF = O_2, \quad FM^+ = M^+ E = O_2, \quad gh^T F = (h^T F)^+ y = O_2, \]

where \( O_2 \) is the zero square matrix of order 2.

Firstly, from (2.2), \( E = I_2 - MM^+ \), \( F = I_2 - M^* M \), we have

\[ EM = (I_2 - MM^+) M = M - MM^+ M = O_2, \quad FM^+ = (I_2 - M^* M)^+ = M^+ = M^+ M^* M = M^+ (I_2 - MM^+) = M^+ E, \]

\[ gh^T F = (1 - h^T F)^{-1} h^T F = h^T F, \quad (h^T F)^+ = (h^T F)^+ (h^T F)^+ = O_2, \]

The group inverse of a \( 3 \times 3 \) matrix over ring with unity 1.

In the notation (2.2), we assume both \( Ev \) and \( h^T F \) to be regular elements in \( R^{2 \times 2} \) and \( R^{2 \times 1} \) respectively. Set

\[ x = 1 - (Ev)^+ (Ev), \quad y = 1 - h^T F (h^T F)^+, \quad w = xx y, \quad (3.1) \]

for fixed but arbitrary \( (Ev)^+ \) and \( (h^T F)^+ \), and \( w \) defined in (2.2). By direct computation, we see that \( xx = x \) and \( yy = y \).

The von Neumann regularity of the matrix \( Z \in R^{3 \times 3} \) defined as in (2.3) is characterized in terms of the regularity of \( w \) as an element of the ring \( R \) in our next lemma. A representation for a \([1]\)-inverse of \( A \) when it exists will prove extremely useful in the solution to our problem.

\[ \square \]

**Lemma 3.** Let \( E,F,S,x,y \) and \( w \) be as in (2.2) and (3.1). We have that \( Z = \begin{bmatrix} s & h^T F \\ Ev & M \end{bmatrix} \) is regular in \( R^{3 \times 3} \) if and only if \( w \) is regular in \( R \).

Proposition 2.4. Let \( X,Y,Z \in R^{3 \times 3} \). If \( N = XYZ \)

where \( X \) and \( Y \) are units and \( Z \) is regular, then the group inverse of \( N \) exists if and only if \( T = ZYX + I_2 - ZZ^+ \) is a unit of \( R^{3 \times 3} \), independent of the choice of \( Z^- \). Equivalently, \( S = YXZ + I_2 - Z \).

Now, consider
Let us denote

\[ PZQ := \begin{bmatrix} 1 -ys(Ev)^t & s \ h^t F \\ 0 & I_2 \end{bmatrix} E \begin{bmatrix} 1 & 0^t \\ -F(h^t F)^t s & I_2 \end{bmatrix} = \begin{bmatrix} s -ys(Ev)^t EEv \ h^t F - ys(Ev) \ EM^t \\ Ev \ M \end{bmatrix} \begin{bmatrix} 1 & 0^t \\ -F(h^t F)^t s & I_2 \end{bmatrix} = \begin{bmatrix} s -ys(Ev)^t Ev -h^t F(h^t F)^t s + ys(Ev)0F(h^t F)^t s \ h^t F - ys(Ev)0 \\ Ev -0(h^t F)^t s & M \end{bmatrix} = \begin{bmatrix} s -s(Ev)^t Ev +h^t F(h^t F)^t s(Ev)^t Ev -h^t F(h^t F)^t s \ h^t F \\ Ev -Ev -0(h^t F)^t s & M \end{bmatrix} \]

\[ = \begin{bmatrix} s \begin{bmatrix} 1 -h^t F(h^t F)^t \ xx -h^t F(h^t F)^t s \ h^t F \\ Ev -Ev -0(h^t F)^t s & M \end{bmatrix} \begin{bmatrix} 1 -h^t F(h^t F)^t \ xx -h^t F(h^t F)^t s \ h^t F \\ Ev -Ev -0(h^t F)^t s & M \end{bmatrix} \]

\[ = \begin{bmatrix} 1 -h^t F(h^t F)^t \ xx -h^t F(h^t F)^t s \ h^t F \\ Ev -Ev -0(h^t F)^t s & M \end{bmatrix} \]

From above \[ PZQ = \bar{Z} \], consider \( P \) and \( Q \) are nonsingular matrices. We have

\[ Z = PZQ = \begin{bmatrix} w \ 0 & 0 \\ 0 & O_2 \end{bmatrix} F(h^t F)^t M^t = \begin{bmatrix} w \ 0 \\ O_2 \end{bmatrix} \begin{bmatrix} (EM)^t E \\ F(h^t F)^t M^t \end{bmatrix} = \begin{bmatrix} w & 0 \\ 0 & O_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ O_2 & O_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & O_2 \end{bmatrix}. \]

Therefore \( Z \) is regular if and only if \( \bar{Z} \) is regular.

Using \( w = yxx \) from (3.4), we can write

\[ Z = \begin{bmatrix} 0 & h^t F \\ Ev & M \end{bmatrix} \begin{bmatrix} w & 0^t \\ 0 & O_2 \end{bmatrix} = H + W. \]

Next, we must show that \( \bar{Z} \) is regular if and only if \( \bar{w} \) is regular. We have that \( H \) is regular, which

\[ H^+ = \begin{bmatrix} 0 & (EM)^t E \\ F(h^t F)^t M^t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ O_2 & 0 \end{bmatrix}. \]

Now, if \( \bar{Z} \) is regular, then \( W \) is regular, by Lemma 2.2. This implies that \( \bar{w} \) is regular. Conversely, assume that \( \bar{w} \) is regular. Let

\[ X = \begin{bmatrix} xw \ y & (Ev)^t E \\ F(h^t F)^t & M^t \end{bmatrix}. \]
We claim that $X$ is a $[1]$-inverse of $Z$. Consider,

$$
Z = \begin{bmatrix}
  w & h^T F \\
  Ev & M \\
\end{bmatrix}
\begin{bmatrix}
  xw - w + 1 - x \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  \theta^T \\
  M^* + F(h^T F)^* h^T F \\
\end{bmatrix} = \tilde{Z}.
$$

Now,

$$
XZ = \begin{bmatrix}
  xw - y & (Ev)^* E \\
  F(h^T F)^* M^* & Ev \\
\end{bmatrix}
\begin{bmatrix}
  yx \\
  h^T F \\
\end{bmatrix}
= \begin{bmatrix}
  xw - y & (Ev)^* E \\
  F(h^T F)^* M^* & Ev \\
\end{bmatrix}
\begin{bmatrix}
  yx \\
  h^T F \\
\end{bmatrix}
- \begin{bmatrix}
  xw - w + (Ev)^* Ev \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  \theta^* \\
  M^* + F(h^T F)^* h^T F \\
\end{bmatrix}
= \begin{bmatrix}
  xw - w + 1 - x \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  \theta^* \\
  M^* + F(h^T F)^* h^T F \\
\end{bmatrix}.
$$

Indeed,

$$
ZXZ = \begin{bmatrix}
  yx & h^T F \\
  Ev & M \\
\end{bmatrix}
\begin{bmatrix}
  xw - w + 1 - x \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  \theta^* \\
  M^* + F(h^T F)^* h^T F \\
\end{bmatrix} = Z.
$$

In view of (3.2) we conclude that a $[1]$-inverse of $\tilde{Z}$ is given by (3.3) and, thus, $Z$ is regular. It remains to prove (3.4) but the proof of this is straightforward. In fact, for $Z$ in (2.3) we have

$$
Z = \begin{bmatrix}
  1 & 0 \\
  -F(h^T F)^* I_z & -F(h^T F)^* M^* \\
\end{bmatrix}
\begin{bmatrix}
  xw - (Ev)^* E \\
  0 \\
\end{bmatrix}
= \begin{bmatrix}
  1 & -Ev(Ev)^* E \\
  0 & (1 - Ev(Ev)^*) E \\
\end{bmatrix}.
$$

(3.3)

is a $[1]$-inverse of $Z$. By direct computation we have

$$
I_z - ZZ^* = \begin{bmatrix}
  (1 - ww^*) \gamma & -(1 - ww^*) \gamma(1 - Ev(Ev)^* E) \\
  0 & 1 - Ev(1 - Ev(Ev)^*) E \\
\end{bmatrix}.
$$

(3.4)
From, our assumption, we have \( M \) is the group inverse. Then we can set \( M^* = M^\dagger \). In this case, in the notation of (2.2) we have 
\[
E = I_2 - MM^* = I_2 - M^*M = F.
\]
It follows that 
\[
(M + E)^{-1} = M^\dagger + E.
\]

**Theorem 3.2.** Let \( M \) be group invertible. With the notation (2.2) and under the assumptions of (3.1), with \( M^\dagger \) replaced by \( M^* \) if \( \omega \) is regular in \( R \) then the group inverse of the matrix 
\[
N = \begin{bmatrix}
e & h_1 & h_2 \\
v_1 & a & c \\
v_2 & b & d
\end{bmatrix} = \begin{bmatrix}e & h^T \\
v & M
\end{bmatrix}
\]
exists if and only if
\[
T = I_2 + x(M^*vMv + M^*Ev) - (vMv + M^*Ev)\gamma - \gamma(M^*vMv + M^*Ev)\gamma - \gamma
\]
is a unit of \( R \). In this case, 
\[
N^\dagger = \begin{bmatrix}
o_1 & \Omega_1 \\
o_2 & \Omega_2
\end{bmatrix}
\]
where 
\[
o_1 = \gamma - x(\gamma - \gamma^2), \\
o_2 = -M^*(vMv + M^*Ev)\gamma + (M^*vMv + M^*Ev)\gamma - \gamma
\]
with
\[
\gamma = r^{-1}(1 - \omega\omega), \\
\Theta = r h^T M^* + r h^T E, \\
\lambda = \gamma(1 + h^T (M^*)^2 (vMv + M^*Ev)) + r^{-1}(1 + h^T (M^*)^2 (vMv + M^*Ev))\gamma.
\]

**Proof:** Write \( N = XZY \) as in (2.3), by Proposition 2.4, the group inverse of \( N \) exists if and only if 
\[
T = I_2 + x(M^*vMv + M^*Ev) - (vMv + M^*Ev)\gamma - \gamma
\]
is a unit, independent of the choice of \( Z \). For the \( \{1\} \)-inverse provided in Lemma 3.1, we have 
\[
T = I_2 + x(M^*vMv + M^*Ev) - (vMv + M^*Ev)\gamma - \gamma
\]
is a unit of \( R \). In this case, 
\[
N^\dagger = \begin{bmatrix}
o_1 & \Omega_1 \\
o_2 & \Omega_2
\end{bmatrix}
\]
where 
\[
o_1 = \gamma - x(\gamma - \gamma^2), \\
o_2 = -M^*(vMv + M^*Ev)\gamma + (M^*vMv + M^*Ev)\gamma - \gamma
\]
with
\[
\gamma = r^{-1}(1 - \omega\omega), \\
\Theta = r h^T M^* + r h^T E, \\
\lambda = \gamma(1 + h^T (M^*)^2 (vMv + M^*Ev)) + r^{-1}(1 + h^T (M^*)^2 (vMv + M^*Ev))\gamma.
\]
We have

\[ TG = \begin{bmatrix} s + (1 - w^w)_y & u^T \\ v & K \end{bmatrix}. \tag{3.8} \]

where

\[ K = M + E + MM^\delta(Ev)^y E = (M + E)(1 + MM^\delta(Ev)^y E), \]

\[ u = (s + (1 - w^w)_y)(ev)^y E + (sh^T M^\delta + h^T E - (1 - w^w)ys(Ev)^y E). \tag{3.9} \]

Since the element \( M^\delta(Ev)^y E \) is \( \mathbb{R} \) nilpotent, because

\[ (M^\delta(Ev)^y E)(M^\delta(Ev)^y E) = M^\delta(Ev)^y E(M^\delta(Ev)^y E) = M^\delta(Ev)^y E = (1-w^w)_y(Ev)^y E = 0. \]

It follows that \( 1 - M^\delta(Ev)^y E \) is a unit of \( \mathbb{R} \).

\[ (1-M^\delta(Ev)^y E)(1-M^\delta(Ev)^y E) = ((1-M^\delta(Ev)^y E) + M^\delta(Ev)^y E(1-M^\delta(Ev)^y E)) = 1 = M^\delta(Ev)^y E + M^\delta(Ev)^y E - M^\delta(Ev)^y EEM^\delta(Ev)^y E \]

Moreover \( M + E \) is a unit because \( M \) has group inverse. Thus, \( K \) is a unit and

\[ S = K^{-1} = (1 - M^\delta(Ev)^y E)(M^\delta + E). \tag{3.10} \]

On account that the element \((2,2)\) of the matrix \( TG \) is a unit, if and only if the Schur complement is a unit of \( R \), it follows that \( TG \) is a unit of \( R^{2 \times 2} \). Therefore, the matrix \( T \) is a unit if and only if

\[ r = s + (1 - w^w)_y - uSv \tag{3.11} \]

is a unit in \( \mathbb{R} \). From (3.10), we get \( Sv = Ev + M^\delta vx \).

Now,

\[ S\delta = [(1 - M^\delta(Ev)^y E)(M^\delta + E)]v \]

\[ = [(M^\delta + E - M^\delta(Ev)^y E)(M^\delta + E)]v \]

\[ = (M^\delta + E - M^\delta(Ev)^y EM^\delta(Ev)^y E)v \]

\[ = M^\delta + Ev - M^\delta(Ev)^y EEv \]

\[ = Ev + M^\delta v - M^\delta(Ev)^y EEv \]

\[ = Ev + M^\delta v(1 - (Ev)^y Ev) \]

\[ = Ev + M^\delta vx. \]

Further, using that last relation of (3.9) we obtain

\[ uSv = x(1 - x) + h^T Ev - (1 - w^w)_x[(s - 1) + (1 + h^T (M^\delta(Ev)^y)^2)v]. \]

By substituting this expression in (3.11), we conclude that \( r \) has the form given in (3.5). By Proposition 2.4, \( N^\delta = XT^{-2}ZY \). From (3.8), it follows that
\[
(TG)^{-1} = \begin{bmatrix}
    r^{-1} & -r^{-1} u S \\
    -S vr^{-1} & S + S vr^{-1} u S
\end{bmatrix}
\]

Next, we compute
\[
XT^{-1} = XG(TG)^{-1} = \begin{bmatrix}
    1 & h^T M^# \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    1 & (Ev)^* E - h^T M^# \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r^{-1} & -r^{-1} u S \\
    -S vr^{-1} & S + S vr^{-1} u S
\end{bmatrix}
\]
\[
= \begin{bmatrix}
    x r^{-1} (Ev)^* E - x r^{-1} u S \\
    -S vr^{-1} & S + S vr^{-1} u S
\end{bmatrix}
\]

Now,
\[
XT^{-1} = XG(TG)^{-1} = \begin{bmatrix}
    1 & h^T M^# \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    1 & (Ev)^* E - h^T M^# \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r^{-1} & -r^{-1} u S \\
    -S vr^{-1} & S + S vr^{-1} u S
\end{bmatrix}
\]
\[
= \begin{bmatrix}
    1 & (Ev)^* E - h^T M^# + h^T M^# \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r^{-1} & -r^{-1} u S \\
    -S vr^{-1} & S + S vr^{-1} u S
\end{bmatrix}
\]
\[
= \begin{bmatrix}
    1 & (Ev)^* E \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r^{-1} & -r^{-1} u S \\
    -S vr^{-1} & S + S vr^{-1} u S
\end{bmatrix}
\]
\[
= \begin{bmatrix}
    1 & (Ev)^* E \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r^{-1} & -r^{-1} u S + (Ev)^* ES + (Ev)^* S vr^{-1} u S \\
    0 & 1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
    r^{-1} & -r^{-1} u S + (Ev)^* ES + (Ev)^* S vr^{-1} u S \\
    0 & 1
\end{bmatrix}
\]

the last equality is due to the fact that \( ES = E \). In the sequel, we denote
\[
\eta = r^{-1}(1 - w r^{-1}) y \quad \text{and} \quad \Theta = y h^T M^# + r^{-1} h^T E.
\]

From (3.10), (3.9), and (3.11) it follows that
\[
SM = M \Theta M, \quad r^{-1} u h M^# M = -y h^T M^# \
\]
\[
r^{-1} u S v = r^{-1} (s + (1 - w r^{-1}) y) - 1 = r^{-1} s + \eta - 1
\]
respectively. In deriving the last equality, we have multiplied on the left expression (3.11) by \( r^{-1} \). Then
Using (3.12) and (3.14), we obtain

Using this latter expression and (3.13) we get

By substituting this into (3.15), using \( r^{-1}uM^* = -h^T(M^*)^2 \), and regrouping terms we get (3.6).

**Conclusions**

In this work we have studied conditions for the existence of the group inverse of the \( 3 \times 3 \) matrix over a ring with unity \( 1 \) under some development certain conditions.

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References


