The Detection of Shifts in the Ratio of Two Poisson Means with an Exponentially Weighted Moving Average Control Chart

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Abstract

The Exponentially Weighted Moving Average (EWMA) control chart is widely used for detecting and controlling the variations in production processes in order to gain efficient manufacturing process. The control chart can be applied to engineering, medical, financial, psychology fields, and etc. In general, one of the control chart performance metrics is Average Run Length (ARL). Therefore, the objective for this research is to approximate the ARL using Markov Chain Approach (MCA) for EWMA control charts for a binomial distribution underlying the ratio of two Poisson means. The proposed MCA is compared with Monte Carlo Simulation (MC) by using absolute percentage relative error. In addition, this research compared the efficiency of EWMA and Cumulative Sum (CUSUM) control charts and found that in the aspect of process change detection, CUSUM performed better than EWMA for all changing levels.

Keywords: Exponentially weighted moving average, Cumulative Sum, average run length

Introduction

A Statistical Process control Charts (SPC) is a control chart for controlling the manufacturing process in order to have a constant production rate or adapt the production process for better outputs. In addition, it helps decreasing the variations in manufacturing process such that the product quality standards are met. In order to increase the ability in checking the quality control with faster and less errors, for example in 1924, Shewhart from USA proposed a well-known control chart, called Shewhart control chart, which was the first control chart to apply for a production process. The control chart was able to detect large changes $(\delta > 1.5\sigma)$. In 1954, a British statistician proposed a Cumulative Sum Control Chart (CUSUM) and in 1959, Robert proposed an Exponentially Weighted Moving Average Control Chart (EWMA). Both control charts detect small changes $(\delta < 1.5\sigma)$ better than the Shewhart control chart. In practice, EWMA control chart is widely used for detecting the changing means and variations in the performance of a stochastic system. In addition, Nong, Connie, and Yebin (2002) applied the EWMA control chart for detecting the changes in the intensity of events for network intrusion detection systems. Han, Tsung, Li, and Xian (2010) studied the EWMA and CUSUM control charts in economics and finances for detecting the turning point in IBM's stocks. Zhang and Chen (2004) proposed to use EWMA control chart for life time data.

Generally, count data was described using a Poisson model such as the number of non-conforming items in an inspection unit, number of accidents among factory workers, the production quality measurement from blemishes, number of patients monthly in a clinic. The suitable distribution for these data is Poisson distribution. Control Charts for Attributes is a control chart for detecting the characteristics of products, such as good or bad, normal or damage, blemish or non-blemish. The widely used Control Charts for Attributes are p chart, c chart, and u chart. Literature is also available on control charts based on counted data. Brook and Evans (1972) studied CUSUM chart for monitoring the mean of a Poisson process. Lucas (1985) described the design and implementation procedure for counted data for detection of increase or decrease in the count level. Gan (1990) worked on the Average Run Length for the EWMA chart for Poisson data. Recently Chen, Zhou, Chang, and Huang (2008) obtained attribute control charts using a generalized zero-inflated Poisson distribution. However, in many situations the traditional technique of the Shewhart control chart may not be suitable, as for many processes, the assumptions of a Poisson distribution may provide an inadequate model. Distribution of counts generated by various types of processes cannot be modeled by the Poisson distribution for use in a c-chart. Recently, Hoffman (2003) developed control limits based on negative binomial for counted data with extra Poisson variation.

The problem of comparing two Poisson rates has been studied in the literature, for example, Nelson (1987); Sahai and Misra (1992); Price and Bonett (2000); Bratcher and Stamey (2004), Krishnamoorthy and Thompson (2004); and Gu, Ng, Tang, and Schucany (2008). In many cases, a binomial distribution may be more flexible and natural to use in place of a Poisson distribution, when the control charts for the ratio of two Poisson means need to be constructed, as a situation may require controlling the ratio rather than a single parameter. In such situation, a binomial distribution derived based on the ratio of two Poisson means can be used to develop the EWMA chart.

Average Run Length (ARL) is a metric to measure the performance of a control chart whether a process is in-control or out-of-control. ARL is the number of average samples in the control limit before the process signals the out-of-control for the first time. ARL can be categorized into 2 states, ARL_0 (a process is in-control) and ARL_1 (a process is out-of-control). For a good performance control chart, the value of ARL_1 should be small.

In this paper, we approximate the Average Run Length (ARL) using the Markov Chain Approach (MCA) method of the EWMA chart when observations are a binomial distribution underlying the ratio of two Poisson means. The rest of this paper is organized as follows. In the next section we give a description of the characteristics of control charts. The approximation of the ARL using the Markov Chain Approach is presented in Section 3. The numerical results are reported in Section 4 and Section 5 presents the conclusion.

Methods and Materials

1. Characteristics of Control Charts

Given an sequence $X_1, X_2, ..., X_k$ be independent identically distributed random variables. Each random variable is generated using a binomial distribution $F(x,n,p)$, where p is a provided parameter. Based on the assumption that an in-control condition the parameter is known (i.e., $p = p_0$), at some change-point time $\nu <$ infinity, the parameter p may be changed to an out-of-control where value p is not equal to p_0.

From the previous paragraphs, the performance metric of a control chart is the Average Run Length, which has 2 states, ARL_0 (in-control ARL) and ARL_1 (out-of-control ARL). ARL_0 (in-control ARL) is the average of samples before the process is out-of-control limit. Generally, the ARL_0 should be large. While ARL_1 is the average of samples when the process changes until the process is out of control. Therefore, the ARL_1 should be small. The condition of the stopping time τ is that

$$ARL_0 = E_\infty(\tau) = K \qquad (1)$$

where K is given (usually large) and $E_\infty(.)$ is the expectation under distribution $F(x,n,p)$, for the in-control process. The quantity $E_\infty(\tau)$ is usually called the in-control Average Run Length (ARL_0).

Another characteristic of a control chart is obtained by minimizing the quantity

$$ARL_1 = E_\nu(\tau), \qquad (2)$$

where $E_\nu(.)$ is the expectation under the assumption that the change-point happens at time $\nu < \infty$ and p is the value of the parameter after the change-point. In practice, the condition in Eq. (2) is usually evaluated when $\nu = 1$ as a zero stae. The quantity $E_1(\tau)$ is usually called the out-of-control Average Run Length (ARL_1).

1.1 Binomial model when the underlying distribution is the ratio of two Poisson means

Let X and Z be two independent Poisson variables with parameters α and β respectively. Then, the condition distribution of X given $X+Z$ follows a binomial distribution with parameters n and $p = \frac{\alpha}{\alpha+\beta}$. The binomial underlying the distribution is the ratio of the density function of two Poisson means $X \sim Bi(n, p = \frac{\alpha}{\alpha+\beta})$, which can be written as

$$f(x) = \binom{n}{x}\left(\frac{\alpha}{\alpha+\beta}\right)^x \left(\frac{\beta}{\alpha+\beta}\right)^{n-x} \quad ; \ x = 0,1,2,\ldots,\mathrm{n}. \qquad (3)$$

The mean and variance of the above distribution are as follows

$$E(X) = n\left(\frac{\alpha}{\alpha+\beta}\right)$$

and

$$Var(X) == \frac{n\alpha\beta}{(\alpha+\beta)^2}.$$

1.2 The Exponentially Weighted Moving Average **Control Chart**

The Exponentially Weighted Moving Average (EWMA) control chart was introduced by Roberts (1959) which this chart is an effective alternative to the Shewhart procedure for detecting small changes of a process mean. The EWMA statistics can be shown as follows:

$$Y_i = \lambda X_i + (1-\lambda) Y_{i-1}, \quad i = 1,2,3,\ldots \qquad (4)$$

where λ is the weight of past information, $0 < \lambda < 1$. The control limits of binomial EWMA control chart are

$$UCL_{EWMA} = n\left(\frac{\alpha}{\alpha+\beta}\right) + h_U \sqrt{\frac{n\alpha\beta}{(\alpha+\beta)^2} \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)}$$

and

$$LCL_{EWMA} = n\left(\frac{\alpha}{\alpha+\beta}\right) - h_L \sqrt{\frac{n\alpha\beta}{(\alpha+\beta)^2} \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)},$$

where h_U and h_L are the coefficient of the EWMA control limit to correspond the value of ARL_0 and $i \to \infty$, the control limits of the EWMA chart can be rewritten as follows:

$$UCL_{EWMA} = n\left(\frac{\alpha}{\alpha+\beta}\right) + h_U\sqrt{\frac{n\alpha\beta}{(\alpha+\beta)^2}\frac{\lambda}{2-\lambda}}$$

and

$$LCL_{EWMA} = n\left(\frac{\alpha}{\alpha+\beta}\right) - h_L\sqrt{\frac{n\alpha\beta}{(\alpha+\beta)^2}\frac{\lambda}{2-\lambda}}.$$

1.3 The Cumulative Sum Control Chart

The Cumulative Sum (CUSUM) control chart is designed to detect mean changes of an independent and identical distribution (*i.i.d*) in an observed sequence of random variables. In general, CUSUM statistics can be written as a sequence with

$$Z_i = \max(Z_{i-1} + X_i - a,\ 0), i = 1, 2, ... \qquad (5)$$

where Z_i is the CUSUM statistics, X_i is the sequence of a binomial observation, $Z_0 = u$ is the initial value, and a is the constant recall reference value of the CUSUM control chart. The first passage time of the CUSUM control chart is given by:

$$\tau_b = \inf\{t > 0 : Z_i > b\},$$

where b is the constant parameter known as the upper control limit.

1.4 Comparison of Analytical Results

The efficiency of the MCA method is measured by the absolute percentage of relative error (*Diff* (%)). The ARL values obtained from the MCA are compared with the values obtained from the Monte Carlo simulation (MC) method under the same parameters.

The comparison of efficiency based on the absolute percentage of relative error (*Diff* (%)) is defined as:

$$Diff(\%) = \frac{|H(u) - \tilde{H}(u)|}{H(u)} \times 100\%. \qquad (6)$$

where $H(u)$ is the ARL of the MCA method values,
and $\tilde{H}(u)$ is the ARL of the Monte Carlo simulation method values.

The criteria for consideration are as follows: If the *Diff* (%) is less than 2%, then ARL values from the MCA and the MC methods are similar and in good agreement.

2. Approximation of the ARL using the Markov Chain Approach

The Markov Chain Approach is one of the most effective methods for studying the characteristics of a control chart. This approach has been discussed by many authors (see Brook and Evans (1972). Lucas and Saccucci (1990) introduced the Markov Chain

Approach for approximating ARL i state in an in-control process where they assume that observation x_j; $j = 1, 2, ..., N$ is an in-control state and $j = N+1$ is out-of-control state. The transition probability, P_{ij}, is the probability of moving from state i to state j in one step and is given by

$$P_{ij} = (X_{ij} = x_j \,|\, X_i = x_i).$$

P_{ij} can be replaced to the transition probability matrix $(\mathbf{P})$ and we get the element of the matrix (P_{ij}) as follows

$$\mathbf{P} = \begin{bmatrix} P_{11} & \mathrm{L} & P_{1N} & | & P_{1,\mathrm{N}+1} \\ \mathrm{M} & \mathrm{O} & \mathrm{M} & | & \mathrm{M} \\ P_{N1} & \mathrm{L} & P_{NN} & | & P_{\mathrm{N},\mathrm{N}+1} \\ --- & --- & --- & | & --- \\ 0 & \mathrm{L} & 0 & | & 1 \end{bmatrix} \text{ or } \mathbf{P} = \begin{bmatrix} P_{11} & \mathrm{L} & P_{1(\mathrm{N}+1)} \\ \mathrm{M} & \mathrm{O} & \mathrm{M} \\ P_{(\mathrm{N}+1)1} & \mathrm{L} & P_{(\mathrm{N}+1)(\mathrm{N}+1)} \end{bmatrix} \text{ or } \mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I}_N - \mathbf{R})\mathbf{1}_N \\ \mathbf{0}_N^T & \mathrm{I}_N \end{bmatrix},$$

where $\mathbf{R}$ is the $N \times N$ transition probability matrix among the in-control states, $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{1}_N$ is the $N \times 1$ column vector of ones, $\mathbf{0}$ is the $1 \times N$ row vector of zeros and 1 is the scalar of one. The k stage transition probability matrix P_k is useful for evaluating the ARL because it contains the probability that the chain goes from one state to another state in k steps. This matrix is

$$\mathbf{P}^k = \begin{bmatrix} \mathbf{R}^k & (\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N \\ \mathbf{0}_N^T & \mathrm{I}_N \end{bmatrix}.$$

The vector $(\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N$ is the vector of transition probabilities from state $i < N+1$ to the state $N+1$ in k steps. Hence

$$P(\tau_i \le \mathrm{k} \,|\, X_0 = x_i) = \text{element}[(\mathbf{I} - \mathbf{R}^k)\mathbf{1}_N](\mathrm{i})$$
$$= \mathbf{P}_N^{(\mathrm{i})\mathrm{T}}(\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N$$

where $\mathbf{P}_N^{(\mathrm{i})\mathrm{T}}$ is the initial probability vector with 1 at i^{th} position and 0 otherwise. Then

$$P(\tau_i = \mathrm{k} \,|\, X_0 = x_i) = P(\tau_i \le \mathrm{k} \,|\, X_0 = x_i) - P(\tau_i \le \mathrm{k}-1 \,|\, X_0 = x_i)$$
$$= \mathbf{P}_N^{(\mathrm{i})\mathrm{T}}(\mathbf{R}^{k-1} - \mathbf{R}^k)\mathbf{1}_N \qquad (7)$$

using Equation (7), the ARL can be rewritten as

$$ARL(\mathrm{N}) = \sum_{k=1}^{\infty} k\mathbf{P}_N^{(i)T}(\mathbf{R}^{k-1} - \mathbf{R}^k)\mathbf{1}_N$$
$$= \sum_{k=1}^{\infty} \mathbf{P}_N^{(i)T}\mathbf{R}^{k-1}\mathbf{1}_N$$
$$= \mathbf{P}_N^{(i)T}(\mathbf{I}_N - \mathbf{R})^{-1}\mathbf{1}_N \qquad (8)$$

where $\mathbf{P}_N^{(i)T}$ is a vector with initial probability vector $\left[0, \ \ldots, \ 0, \ 1, \ 0, \ \ldots, \ 0\right]_{1\times N}$, $\mathbf{I}$ is the identity matrix, and $\mathbf{1}$ is the unit vector.

An ARL approximation by use of MCA for monitoring mean shifts of the process is in interval of the lower and upper control limits. The region of the in-control state is divided into n subintervals.

The j^{th} subinterval of upper control limit (U_j), j^{th} subinterval of lower control limit (L_j) and the i^{th} subinterval of midpoint (m_i) are given by

$$U_j = h_L + \frac{j(h_U - h_L)}{n}$$

$$m_i = h_L + \frac{(2i-1)(h_U - h_L)}{2n}$$

$$L_j = h_L + \frac{(j-1)(h_U - h_L)}{n}.$$

Consequently, the transition probability equation (P_{ij}) can be rewritten as

$$P_{ij} = P(L_j \le Z_i \le U_j \,|\, Z_{i-1} = m_i) \tag{9}$$

and substitute the EWMA statistic (Y_t), L_j, U_j and m_i into Equation (9). This transition probability equation is

$$P_{ij} = P(LCL_{EWMA_{Y_i}} < \lambda Y_i + (1-\lambda)EWMA_{Y_{i-1}} < UCL_{EWMA_{Y_i}} \mid EWMA_{Y_{i-1}} = CL_{EWMA_{Y_i}})$$

$$= P\left(\begin{array}{l} h_L + \dfrac{(j-1)(h_U - h_L)}{n} - (1-\lambda)\left(h_L + \dfrac{(2i-1)(h_U - h_L)}{2n} \right) < \lambda Y_i < h_L + \dfrac{j(h_U - h_L)}{n} \\ -(1-\lambda)\left(h_L + \dfrac{(2i-1)(h_U - h_L)}{2n} \right) \end{array} \right)$$

$$= P\left(\begin{array}{l} h_L + \dfrac{(j-1)(h_U - h_L)}{n} - (1-\lambda)\left(h_L + \dfrac{(2i-1)(h_U - h_L)}{2n} \right) < \lambda Y_i < h_L + \dfrac{j(h_U - h_L)}{n} \\ -(1-\lambda)\left(h_L + \dfrac{(2i-1)(h_U - h_L)}{2n} \right) \end{array} \right)$$

$$= P\left(\begin{array}{l} h_L + \dfrac{(h_U - h_L)}{2n\lambda}\big(2(j-1) - (1-\lambda)(2i-1)\big) < Y_i \\ < h_L + \dfrac{(h_U - h_L)}{2n\lambda}\big(2j - (1-\lambda)(2i-1)\big) \end{array} \right).$$

For a one-sided, the transition probability equation is

$$P_{ij} = P\left(\begin{array}{l} \dfrac{h_U}{2n\lambda}\big(2(j-1) - (1-\lambda)(2i-1)\big) < Y_i \\ < + \dfrac{(h_U)}{2n\lambda}\big(2j - (1-\lambda)(2i-1)\big) \end{array} \right). \tag{10}$$

Results and Discussion

In this section, we compared the results obtained for the ARL_0 and ARL_1 from the MCA method with the results of Monte Carlo simulations for the EWMA chart. We also compare the CPU times required to compute the numerical values for ARL_0 and ARL_1. The results are shown in Table 1 for the in-control parameter value $\alpha_0 = 5$, ARL_0 =370 and in Table 2 for an in-control parameter value $\beta_0 = 12$, ARL_0 =370. The numbers in each cell in the table represent the value of the ARL_0 and ARL_1 and in parentheses () is the CPU time for the calculation. The table shows that the results from the MCA method are close to the MC simulation results. Obviously, MCA method for evaluating ARL_0 and ARL_1 is very effective alternatives to the MC. As shown in Tables1 -2, the use of the MCA method for ARL_0 and ARL_1 can greatly reduce the computation times as compared with MC approach. In Tables 3, the ARL_0 and ARL_1 values for the MCA method were calculated from Equation (8). For the EWMA we obtain a pair of optimal parameter values of λ =0.05, h_U =10.989998, for the CUSUM procedure we used a boundary value $a = 10, b = 5.05$. The results in Table 3 show that for all magnitude of changes, CUSUM chart performs better than EWMA chart. Table 4 shows that the results are similar to those in Table 3.

Table 1 Comparison of ARL between the MCA and the MC methods when $\beta_0 = 10, ARL_0 = 370$

α	MCA	MC	*Diff* (%)
	h_U =34.849	h_U =34.849	
5.0	370.244 (158)	369.359 ± 0.105 (3245)	0.239
5.1	192.438 (159)	191.438 ± 0.108 (1241)	0.52
5.3	100.797 (158)	100.373 ± 0.303 (1025)	0.421
5.5	74.959 (159)	74.954 ± 0.145 (874)	0.007
5.7	62.497 (159)	62.590 ± 0.089 (745)	0.149
6.0	52.175 (159)	52.238 ± 0.054 (458)	0.121
6.5	42.729 (158)	42.784 ± 0.034 (351)	0.129
7.0	37.095 (158)	37.126 ± 0.024 (312)	0.084
8.0	30.446 (159)	30.460 ± 0.016 (268)	0.046

The numbers in parentheses () are CPU times in seconds.

Table 2 Comparison of the ARL between the MCA and MC methods when $\alpha_0 = 8, ARL_0 = 370$

β	MCA	MC	*Diff* (%)
	h_U =41.56	h_U =41.56	
12.0	371.264 (158)	369.223 ± 0.126 (4582)	0.549
11.9	265.377 (158)	265.687 ± 0.128 (1856)	0.117
11.7	165.946 (158)	165.791 ± 0.818 (1226)	0.093
11.5	121.942 (159)	120.604 ± 0.438 (1052)	1.097
11.3	99.169 (158)	99.336 ± 0.278 (902)	0.168
11.0	79.862 (159)	80.073 ± 0.162 (821)	0.264
10.5	63.419 (158)	63.385 ± 0.082 (729)	0.054
10.0	53.922 (159)	53.939 ± 0.052 (623)	0.032
9.0	42.469 (158)	42.464 ± 0.028 (421)	0.012

The numbers in parentheses () are CPU times in seconds.

Table 3 Comparison of the ARL of the EWMA and CUSUM charts when $\beta_0 = 9, ARL_0 = 370$

α	EWMA	CUSUM
	$\lambda = 0.05$ h_U = 10.98998	$a = 10$ b = 5.05
1.0	369.256	369.362 ± 0.124
1.1	83.878	59.254 ± 0.245
1.3	40.404	21.742 ± 0.057
1.5	29.212	13.476 ± 0.029
1.7	23.542	9.927 ± 0.018
2.0	18.652	7.261 ± 0.012
2.5	14.267	5.213 ± 0.008
3.0	11.815	4.179 ± 0.006
4.0	9.129	3.142 ± 0.004

Table 4 Comparison of the ARL of the EWMA and CUSUM charts when $\alpha_0 = 10, ARL_0 = 500$

β	EWMA	CUSUM
	$\lambda = 0.05$ h_U = 68.3233	$a = 66.67$ b = 100
5.0	497.695	500.211 ± 1.056
4.9	241.659	183.831 ± 1.093
4.7	118.524	71.415 ± 0.267
4.5	87.649	43.742 ± 0.142
4.3	73.585	31.342 ± 0.078
4.0	61.671	21.841 ± 0.044
3.5	50.407	14.190 ± 0.022
3.0	43.260	10.320 ± 0.013
2.0	33.926	6.535 ± 0.006

Conclusion and Suggestion

In this paper, an approximation of the Average Run Length (ARL) using the Markov Chain Approach of the EWMA control chart with a binomial distribution underlying the ratio of two Poisson means are presented. The ARL obtained by the MCA method is close to the MC simulation results. The MCA method clearly takes much less computational time than the Monte Carlo (MC) simulation method. In addition, the performance of the CUSUM chart is superior to the EWMA chart for all magnitudes of changes when the observation is a binomial distribution underlying the ratio of two Poisson means. The MCA approach can be applied to some other distributions, e.g. a zero inflated Poisson distribution. It is suggested that an approximation of the Average Run Length (ARL) using the Markov Chain Approach of CUSUM chart can be applied to real data, empirical data, and real-world situations applications for a variety of data processes such as in medical and demographic, economics, finance, environmental, etc. These issues should be addressed in future research. Furthermore, MCA method for evaluation ARL can be developed other control charts such as Cumulative sum, Double Exponentially weighted Moving Average control charts, etc.

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