A Noncommutative Semigroup which Contains the Natural Numbers under Addition and Its Left Ideals

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Abstract

In this paper, we introduce a new semigroup which is not commutative. Then we investigate some properties in this structure. We discover that this semigroup has no a proper right ideal. After that we find some subsemigroups of this semigroup which are left ideals. Finally, we prove that the natural numbers under addition can be embedded into this semigroup.

Keyword: Semigroup, Subsemigroup, Left ideal, Right ideal, Isomorphism

Introduction

In mathematics, semigroup $(S,*)$ is an algebraic structure consisting of a nonempty set S together with an associative binary operation $*$ on S, i.e. $(x*y)*z=x*(y*z)$ for every x,y and z in S. As long as not otherwise stated, we write the semigroup operation as a multiplication. And we mostly omit it typographically, i.e. we write S instead of $(S,*)$, xy instead of $x*y$, $x(yz)$ instead of $x*(y*z)$ and soon.

A semigroup, unlike a group, need not have an identity element and its elements need not have inverse within the semigroup. These mean that every group is also a semigroup with the identity. Of course, the notion in an algebraic semigroup theory is a straightforward generalization of a group theory. Thus several researchers are interested in the classes of semigroups.

In fact, the natural number $\mathbb{N}$ is a semigroup under additive operation. This is a good example to increase understanding about the algebraic structure in the semigroup theory. Backelin, (1990) studied the natural numbers under addition into three parts. The first part consists the number of all difference subsemigroups of the natural numbers under addition which corresponding to a given Frobenius number. Moreover, they counted the number of all maximal subsemigroup S with a given Frobenius number. Finally, they found the number of all subsets X of $\{1,2,\dots,n\}$ such that there are at most q different sums of pairs of elements from X. Barucci, (2009) studied deals mainly with numerical semigroups, i.e. subsemigroups of $\mathbb{N}$ with zero and with finite complement in $\mathbb{N}$ and also studied commutative ring theory.

It is easy to see that the natural numbers under addition is commutative and satisfies cancellative property. Therefore, many definitions coincide under this structure. It is well-known that each subsemigroup of a commutative semigroup is still a commutative semigroup. Similarly, every subsemigroup also preserves cancellative property. These motivate us to find a new semigroup which contains the natural numbers under addition and its operation is not commutative.

In this research, we define a new semigroup which covers the natural numbers under addition. Some characterizations in this structure are described. Moreover, we find some left ideals of the noncommutative semigroup and prove that $(\mathbb{Z},+)$ is isomorphic to a subgroup of the noncommutative semigroup.

Preliminaries

In this section, we give some definitions that will be used in this paper.

Definition 1. A semigroup S is said to be a commutative semigroup if the operation is commutative. That is, $ab = ba$ for every $a, b \in S$.

Definition 2. A nonempty subset A of a semigroup S is called a subsemigroup of S if A is closed under the operation, that is, $ab \in A$ for every $a, b \in A$.

Definition 3. An element e of a semigroup S is called a left [right] identity element of S if $ea = a$ $[ae = a]$ holds for every $a \in S$.

Definition 4. An element e of a semigroup S is called an idempotent element if $e^2 = e$.

Definition 5. A semigroup S is called a left [right] cancellative semigroup if $zx = zy$ $[xz = yz]$ implies $x = y$ for every $x, y, z \in S$.

Definition 6. An element a of a semigroup S is called a regular element if there is $x \in S$ such that $a = axa$.

Definition 7. A nonempty subset A of a semigroup S is called a left [right] ideal of S if $SA \subseteq A$ $[AS \subseteq A]$, that is $sa \in A$ $[as \in A]$ for every $a \in A$ and $s \in S$.

Main Results

In this section, we investigate a new semigroup from the power set of the set of all integers. We show that this semigroup is not a commutative semigroup. We find out that this structure does not have a right ideal. Then we find some subsemigroups of this semigroup which are left ideals. Moreover, we prove that the natural number under addition can be embedded into this semigroup.

Lemma 1. Let $A, B \in P(\mathbb{Z})$ be such that A and B have a minimum element. Then

$$\min A + \min B = \min\{\min A + b : b \in B\}.$$

Proof. Suppose that $c = \min A + \min B$ and $C = \{\min A + b : b \in B\}$. We note from $\min B \in B$ that $c = \min A + \min B \in C$. Let x be an arbitary element in C. Then $x = \min A + b$ for some $b \in B$. Since $b \in B$, we have $\min B \leq b$. So

$$c = \min A + \min B \leq \min A + b = x.$$

This implies that $c \leq x$ and we get that $c = \min C$. Hence the proof is completed.

Theorem 2. Let $\mathbb{S} = \{A \in P(\mathbb{Z}) : A$ has a minimum element$\}$. Define $* : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ by

$$A * B = \{\min A + b : b \in B\} \text{ for all } A, B \in \mathbb{S}.$$

Then $(\mathbb{S}, *)$ is a semigroup.

Proof. We can see that $\mathbb{S} \neq \emptyset$ since $\{0\} \in \mathbb{S}$. Let $A, B \in \mathbb{S}$, then $A, B \in P(\mathbb{Z})$ and A, B have a minimum element. To show that $A * B \subseteq \mathbb{Z}$, let $x \in A * B$ and then $x = \min A + b$ for some $b \in B$. As $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Z}$, we have $x = \min A + b \in \mathbb{Z}$. Therefore $A * B \in P(\mathbb{Z})$. We note from A, B have a minimum element and by Lemma 1 that $A * B$ has a minimum element. Thus $A * B \in \mathbb{S}$. Next, we let $A, B, C \in \mathbb{S}$. To show that $(A * B) * C = A * (B * C)$, let $x \in (A * B) * C$. Then $x = \min(A * B) + c$ for some $c \in C$. By Lemma 1, we deduce that

$$\begin{aligned} x &= (\min A + \min B) + c \\ &= \min A + (\min B + c). \end{aligned}$$

From $\min B + c \in B * C$, we have $x \in A * (B * C)$. Therefore $(A * B) * C \subseteq A * (B * C)$. Let $x \in A * (B * C)$. Then $x = \min A + y$ for some $y \in B * C$. By $y \in B * C$, we have $y = \min B + c$ for some $c \in C$. From Lemma 1, we get that

$$\begin{aligned} x = \min A + y &= \min A + (\min B + c) \\ &= (\min A + \min B) + c \\ &= \min(A * B) + c. \end{aligned}$$

Thus $x \in (A * B) * C$. So $(A * B) * C = A * (B * C)$. Hence $(\mathbb{S}, *)$ is a semigroup.

From now on, we refer to $(\mathbb{S}, *)$ as the semigroup which is defined in Theorem 2. The following example shows that this semigroup is not a commutative semigroup.

Example 3. Consider $A = \{5, 6, 7\}$ and let

$B = \{1, 2, 4\}$. It is clear that $A, B \in \mathbb{S}$. We compute $A * B = \{6, 7, 9\}$ and so $B * A = \{6, 7, 8\}$. Then $A * B \neq B * A$. This shows that $\mathbb{S}$ is not a commutative semigroup.

Theorem 4. Let $A \in \mathbb{S}$. Then the following statements are equivalent:

(i) $\min A = 0$.

(ii) A is a left identity.

(iii) A is an idempotent.

Proof. $(i) \Rightarrow (ii)$ Assume that $\min A = 0$. Let $X \in \mathbb{S}$. To show that $A * X = X$, let $x \in A * X$. Then $x = \min A + y$ for some $y \in X$. By our assumption, we get that $x = 0 + y = y \in X$. Thus $A * X \subseteq X$. Let $z \in X$, then $z = 0 + z = \min A + z \in A * X$ and hence $X \subseteq A * X$. Therefore A is a left identity.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Assume that A is an idempotent. That is, $A * A = A$. We will show that $\min A = 0$. Since $A * A = A$ and $\min(A * A) = \min A + \min A$, we have $\min A + \min A = \min A$. Hence $\min A = 0$.

Theorem 5. $\mathbb{S}$ is a left cancellative semigroup.

Proof. Let $A, B, C \in \mathbb{S}$ such that $A * B = A * C$. We must show that $B = C$. Let $b \in B$. Then $\min A + b \in A * B$. Since $A * B = A * C$, we have $\min A + b \in A * C$. Thus $\min A + b = \min A + c$ for some $c \in C$. So $b = c \in C$ and thus $B \subseteq C$. Similarly, we deduce that $C \subseteq B$ whence $B = C$. Therefore $\mathbb{S}$ is a left cancellative semigroup.

Example 6. We note that $A = \{1, 2, 3\}, B = \{3\}$ and $C = \{3, 4, 5\}$ are elements in $\mathbb{S}$. We consider $B * A = \{3 + 1, 3 + 2, 3 + 3\} = \{4, 5, 6\}$ and the set $C * A = \{3 + 1, 3 + 2, 3 + 3\} = \{4, 5, 6\}$. Hence we get $B * A = C * A$ but $B \neq C$. This shows that $\mathbb{S}$ is not a right cancellative semigroup.

Theorem 7. $\mathbb{S}$ is a regular semigroup.

Proof. Let $A \in \mathbb{S}$. We will show $A = A * X * A$ for some $X \in \mathbb{S}$. We choose $X = \{-a\}$ where $a = \min A$. Clearly, $X \in P(\mathbb{Z})$ and $\min X = -a$. These mean that $X \in \mathbb{S}$. Let $c \in A * X * A$. Then there exists $y \in A$ such that

$$\begin{aligned} c &= \min(A * X) + y \\ &= \min A + \min X + y \\ &= a + (-a) + y \\ &= y \in A. \end{aligned}$$

Thus $A * X * A \subseteq A$. Let $c \in A$. We obtain that

$$\begin{aligned} c &= a + (-a) + c \\ &= \min A + \min X + c \\ &= \min(A * X) + c \in A * X * A. \end{aligned}$$

Therefore $A \subseteq A * X * A$. So $A = A * X * A$. Since A is an arbitrary element of $\mathbb{S}$, we have $\mathbb{S}$ is a regular semigroup.

Theorem 8. $\mathbb{S}$ has no proper right ideals.

Proof. Let R be any right ideal of $\mathbb{S}$. So $R * \mathbb{S} \subseteq R$. We will show that $R = \mathbb{S}$. Let $A \in \mathbb{S}$. We will verify that $A \in R$. Since $R \neq \emptyset$, so choose $B \in R$, we have $B * A \in R * \mathbb{S} \subseteq R$. We choose $C = \{-\min B - \min A + a : a \in A\}$. Clearly, $C \in \mathbb{S}$. We consider

$$\begin{aligned} (B * A) * C &= \{\min(B * A) + c : c \in C\} \\ &= \{\min B + \min A + c : c \in C\} \\ &= \{\min B + \min A + (-\min B - \\ &\quad \min A + a) : a \in A\} = A. \end{aligned}$$

Note that $A = (B * A) * C \in R * \mathbb{S} \subseteq R$. Therefore $\mathbb{S} \subseteq R$. As $R \subseteq \mathbb{S}$, we get that $R = \mathbb{S}$. Hence $\mathbb{S}$ has no a proper right ideal.

We note from Theorem 8 that the semigroup $\mathbb{S}$ has no any right ideal. Then we will study subsemigroups of $\mathbb{S}$ which are left ideals.

Theorem 9. Let $\mathbb{S}_1 = \{A \in P(\mathbb{Z}) : A$ is a nonempty finite set$\}$. Then $\mathbb{S}_1$ is a left ideal of $\mathbb{S}$.

Proof. We will show that $\mathbb{S} * \mathbb{S}_1 \subseteq \mathbb{S}_1$. Let $A \in \mathbb{S}_1$ and $X \in \mathbb{S}$. Then $A \in P(\mathbb{Z})$ and A is a nonempty finite set. We can write that $A = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and $n \in \mathbb{N}$. This implies that

$$\begin{aligned} X * A = \{&\min X + a_1, \min X + a_2, \\ &\min X + a_3, \ldots, \min X + a_n\} \end{aligned}$$

So $|X * A| = n$. Thus $X * A$ is a nonempty finite set. Therefore $X * A \in \mathbb{S}_1$ and We conclude that $\mathbb{S}_1$ is a left ideal of $\mathbb{S}$.

Corollary 10. Let $\mathbb{S}_1 = \{A \in P(\mathbb{Z}) : A$ is a nonempty finite set$\}$. Then $(\mathbb{S}_1,*)$ is a subsemigroup of $(\mathbb{S},*)$.

Theorem 11. Let $\mathbb{S}_2 = \{A_n : n \in \mathbb{Z}\}$ where $A_n = \{x \in \mathbb{Z} : x \geq n\}$. Then $\mathbb{S}_2$ is a left ideal of $\mathbb{S}$.

Proof. Let $X \in \mathbb{S}$ and $A_n \in \mathbb{S}_2$ where $n \in \mathbb{Z}$. Then $A_n = \{n, n+1, n+2, \ldots\}$. This implies that

$$\begin{aligned} X * A_n &= \{\min X + n, (\min X + n) + 1, \\ &\quad (\min X + n) + 2, \ldots\} \\ &= A_m \end{aligned}$$

where $m = \min X + n$. Therefore $\mathbb{S} * \mathbb{S}_2 \subseteq \mathbb{S}_2$ and so $\mathbb{S}_2$ is a left ideal of $\mathbb{S}$.

Corollary 12. Let $\mathbb{S}_2 = \{A_n : n \in \mathbb{Z}\}$ where $A_n = \{x \in \mathbb{Z} : x \geq n\}$. Then $\mathbb{S}_2$ is a subsemigroup of $\mathbb{S}$.

Theorem 13. Let $\mathbb{S}_2 = \{A_n : n \in \mathbb{Z}\}$ where $A_n = \{x \in \mathbb{Z} : x \geq n\}$. Then $\mathbb{S}_2$ is an abelian subgroup of $\mathbb{S}$.

Proof. It follows from Corollary 12 that $\mathbb{S}_2$ is a subsemigroup of $\mathbb{S}$. We note that $A_0 \in \mathbb{S}_2$. Let $A_n \in \mathbb{S}_2$ where $n \in \mathbb{Z}$. From the proof of Theorem 11, we have $A_0 * A_n = A_{0+n} = A_n$ and hence $A_n * A_0 = A_{n+0} = A_n$. Therefore A_0 is the identity. Since $n \in \mathbb{Z}$, we have $A_{-n} \in \mathbb{S}_2$. This implies that

$$A_n * A_{-n} = A_{n+(-n)} = A_0 = A_{(-n)+n} = A_{-n} * A_n.$$

So A_{-n} is an inverse of A_n. Hence $\mathbb{S}_2$ is a subgroup of semigroup $\mathbb{S}$. Clearly, $A_n * A_m = A_m * A_n$ for all $n, m \in \mathbb{Z}$. Therefore $\mathbb{S}_2$ is an abelian subgroup of $\mathbb{S}$.

For a mapping α, we write $x\alpha$ instead of the image of x under α where x belongs to domain of α. We recall the notion of isomorphism. Let $(S,\cdot)$ and $(S',*)$ be two semigroups. A function $\alpha : S \to S'$ is called a homomorphism if for any $a, b \in S$, $(a \cdot b)\alpha = a\alpha * b\alpha$.

A semigroup S is said to be isomorphic to a semigroup S' if there exists a bijective homomorphism from S onto S' and a semigroup S is embedded into a semigroup S' if there exists an injective homomorphism from S into S'.

Theorem 14. Let $\mathbb{S}_2 = \{A_n : n \in \mathbb{Z}\}$ where $A_n = \{x \in \mathbb{Z} : x \geq n\}$. Then $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{S}_2,*)$.

Proof. Define a homomorphism $\alpha : \mathbb{Z} \to \mathbb{S}_2$ by

$$n\alpha = A_n \text{ for all } n \in \mathbb{Z}.$$

Firstly, we will show that α is a bijection. Let $n, m \in \mathbb{Z}$ be such that $n\alpha = m\alpha$. We get that $A_n = A_m$ and so $n = \min A_n = \min A_m = m$. Thus α is injective. Clearly, α is surjective. Finally, we will verifly that α is a homomorphism. Let $x, y \in \mathbb{Z}$. This implies that

$$(x + y)\alpha = A_{x+y} = A_n * A_m = x\alpha * y\alpha.$$

We obtain that α is an isomorphism. Hence $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{S}_2,*)$.

In consequence of the above theorem, we will deduce that $(\mathbb{N}, +)$ is embedded into $(\mathbb{S},*)$ in the following corollary.

Corollary 15. $(\mathbb{N}, +)$ can be embedded into $(\mathbb{S},*)$.

Let (X, d) and (Y, d_1) be metric spaces. Then (X, d) is isometric to (Y, d_1) if there exists a surjective mapping $\alpha : X \to Y$ such that for all x, y in X, $d(x, y) = d_1(x\alpha, y\alpha)$. Such a mapping α is said to be an isometry. We note that each isometry is an injection. It is well-known that for a nonempty subset X of $\mathbb{Z}$, $(X, |\cdot|)$ is a metric space where $|\cdot|$ is the absolute-value metric.

Proposition 16. Let $A, B \in \mathbb{S}$ and α a surjective isometry from $(A, |\cdot|)$ onto $(B, |\cdot|)$. Then the following statements hold:

(i) if A is a finite set, then $(\min A)\alpha = \min B$ or $(\min A)\alpha = \max B$.

(ii) if A is an infinite set, then $(\min A)\alpha = \min B$.

Proof. Assume that A is a finite set. Put $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$ for some $n \in \mathbb{N}$. Similarly, $B = \{b_1, b_2, \ldots, b_n\}$ where $b_1 < b_2 < \cdots < b_n$. We will show that $(\min A)\alpha =$

$\min B$ or $(\min A)\alpha = \max B$. From the surjectivity of α, there exist $x, y \in A$ such that $x\alpha = b_1$ and $y\alpha = b_n$. Since α is an isometry function, we have $|x - a_1| = |x\alpha - a_1\alpha|$ and $|y - a_1| = |y\alpha - a_1\alpha|$. We conclude from $a_1 = \min A$ and $b_1 = \min B$ that $x - a_1 = a_1\alpha - b_1$ and so $y - a_1 = b_n - a_1\alpha$. These imply that

$$\begin{aligned} x - y &= (x - a_1) - (y - a_1) \\ &= (a_1\alpha - b_1) - (b_n - a_1\alpha) \\ &= 2a_1\alpha - b_1 - b_n. \end{aligned} \tag{3.1}$$

From α is an isometry mapping, we obtain that

$$\begin{aligned} |x - y| &= |x\alpha - y\alpha| \\ &= |b_1 - b_n| \\ &= b_n - b_1. \end{aligned} \tag{3.2}$$

Hence we have two cases to consider;

Case1 : $|x - y| = x - y$. From equation 3.1 and 3.2, we deduce that

$$\begin{aligned} 2a_1\alpha &= (2a_1\alpha - b_1 - b_n) + b_1 + b_n \\ &= (x - y) + b_1 + b_n \\ &= (b_n - b_1) + b_1 + b_n \\ &= 2b_n. \end{aligned}$$

Case2 : $|x - y| = y - x$. From equation 3.1 and 3.2, we get that

$$\begin{aligned} 2a_1\alpha &= (2a_1\alpha - b_1 - b_n) + b_1 + b_n \\ &= (x - y) + b_1 + b_n \\ &= -(b_n - b_1) + b_1 + b_n \\ &= 2b_1. \end{aligned}$$

It follows from two cases that $(\min A)\alpha = \min B$ or $(\min A)\alpha = \max B$.

Assume that A is an infinite set. We will show that $(\min A)\alpha = \min B$. Let $a = \min A$ and $b = \min B$. Since α is a surjection, there exists $x \in A$ such that $x\alpha = b$. From α is an isometry function, we get $|x - a| = |x\alpha - a\alpha| = |b - a\alpha|$. We conclude from the minimality of a and b that

$$x - a = a\alpha - b. \tag{3.3}$$

If $x = \max A$, then A is a finite set which is a contradiction. Hence $x \neq \max A$, there exists $y \in A$ such that $y > x$. We note from α is an isometry function, we get that

$$\begin{aligned} y - x &= |y - x| \\ &= |y\alpha - x\alpha| \\ &= |y\alpha - b| \\ &= y\alpha - b. \end{aligned} \tag{3.4}$$

From equation 3.3 and 3.4, we deduce that

$$\begin{aligned} y - a &= (x - a) + (y - x) \\ &= (a\alpha - b) + (y\alpha - b) \\ &= a\alpha + y\alpha - 2b. \end{aligned} \tag{3.5}$$

Similarly, we conclude from α is an isometry function that

$$\begin{aligned} y - a &= |y - a| \\ &= |y\alpha - a\alpha|. \end{aligned} \tag{3.6}$$

Suppose that $|y\alpha - a\alpha| = a\alpha - y\alpha$. From equation 3.5 and 3.6, we deduce that

$$\begin{aligned} 2y\alpha &= y\alpha + (y\alpha + a\alpha - 2b) - a\alpha + 2b \\ &= y\alpha + (a\alpha - y\alpha) - a\alpha + 2b \\ &= 2b. \end{aligned}$$

Therefore $y\alpha = x\alpha$, then $|x - y| = |x\alpha - y\alpha| = 0$ and so $x = y$ which is a contradiction. So $|y\alpha - a\alpha| = y\alpha - a\alpha$. From equation 3.5 and 3.6, we obtain that

$$\begin{aligned} 2a\alpha &= a\alpha + (a\alpha + y\alpha - 2b) - y\alpha + 2b \\ &= a\alpha + (y\alpha - a\alpha) - y\alpha + 2b \\ &= 2b. \end{aligned}$$

It follows that $(\min A)\alpha = \min B$.

Remark 17. For all $A \in \mathbb{S}$, we note that $A = \{0\} * A \in \mathbb{S} * A$. Hence $\mathbb{S} * A \cup \{A\} = \mathbb{S} * A$. It follows that the principal left ideal generated by A is $\mathbb{S} * A$ and denoted by L_A.

Proposition 18. For any $A \in \mathbb{S}$, let $X, Y \in L_A$. Then $\min X = \min Y$ if and only if $X = Y$.

Proof. Assume that $\min X = \min Y$. Note that $X = B * A$ and $Y = C * A$ for some $B, C \in \mathbb{S}$. We obtain that

$$\begin{aligned} \min B &= \min B + \min A - \min A \\ &= \min(B * A) - \min A \\ &= \min(C * A) - \min A \\ &= \min C. \end{aligned}$$

To verify that $X = Y$, let $z \in X$. Then we have $z = \min B + a$ for some $a \in A$. It follows that $z = \min B + a = \min C + a \in Y$. Therefore $X \subseteq Y$. Similarly, we get that $Y \subseteq X$. Hence $X = Y$.

From the fact that $\min(X * Y) = \min(Y * X)$ and Proposition 18, we obtain the below corollary.

Corollary 19. Let $A \in \mathbb{S}$. Then L_A is a commutative semigroup.

Theorem 20. Let A be arbitrary fixed element in $\mathbb{S}$. Then $(L_A, *)$ is an abelian subgroup of $(\mathbb{S}, *)$.

Proof. From Corollary 19, we obtain that L_A is a commutative semigroup. Let $O = \{-\min A\} * A$, then $O \in L_A$. To show that O is the identity of L_A, let $B \in L_A$. Then $B = C * A$ for some $C \in \mathbb{S}$. Consider,

$$\begin{aligned}\min(B * O) &= \min B + \min O. \\ &= \min B + 0 = \min B.\end{aligned}$$

From Proposition 18, we get that $B * O = B$. Since L_A is a commutative semigroup, we obtain $O * B = B$. So O is the identity of L_A. Put $B' = \{-\min A - \min B\} * A$. It is clear that $B' \in L_A$. Consider,

$$\begin{aligned}\min(B * B') &= \min B + \min B' \\ &= \min B - \min A - \min B + \min A \\ &= 0 = \min O.\end{aligned}$$

We note from Proposition 18 that $B * B' = O$. Since L_A is a commutative semigroup, we deduce that B' is an inverse of B. Hence $(L_A,*)$ is a subgroup of $(\mathbb{S},*)$.

Theorem 21. Let $A \in \mathbb{S}$. Then $(L_A,*)$ is isomorphic to $(\mathbb{Z},+)$.

Proof. Define a homomorphism mapping $\varphi : L_A \to \mathbb{Z}$ by $B\varphi = \min B$ for all $B \in L_A$. Firstly, we will verify that φ is a bijective function. Let $X, Y \in L_A$ be such that $X\varphi = Y\varphi$. So $\min X = \min Y$. By Proposition 18, we have $X = Y$. We have φ is injective. Next, let $n \in \mathbb{Z}$ and put $B = \{-\min A + n\} * A$. Then $B \in L_A$. Consider,

$$B\varphi = \min B = -\min A + n + \min A = n.$$

Then we get that φ is surjective. Clearly, φ is a homomorphism and hence $(L_A,*)$ is isomorphic to $(\mathbb{Z},+)$.

Corollary 22. $(\mathbb{N},+)$ can be embedded into $(L_A,*)$.

Theorem 23. Let $A \in \mathbb{S}$. Then $B \in L_A$ if and only if there exists a surjective mapping $\alpha : B \to A$ such that α is an isometry function and $(\min B)\alpha = \min A$.

Proof. Assume that $B \in L_A$. Then $B = C * A$ for some $C \in \mathbb{S}$ and we let $\min C = c$. Define a mapping α by $b\alpha = b - c$ for all $b \in B$. To show that $\alpha : B \to A$, let $b \in B$. So $b = c + a$ for some $a \in A$. We have

$$b\alpha = b - c = (c + a) - c = a \in A.$$

Therefore $\alpha : B \to A$. Let $x \in A$ and we choose $y = c + x \in C * A = B$. We obtain that

$$y\alpha = y - c = (c + x) - c = x.$$

So α is surjective. Next, we will show that α is an isometry function. Let $x, y \in B$. Then

$$\begin{aligned}|x - y| &= |x - y - c + c| \\ &= \left|(x - c) - (y - c)\right| \\ &= |x\alpha - y\alpha|.\end{aligned}$$

Hence α is an isometry function. Finally, we must show that $(\min B)\alpha = \min A$. We consider

$$\begin{aligned}(\min B)\alpha &= \min B - c \\ &= \min(C * A) - C \\ &= (c + \min A) - c \\ &= \min A.\end{aligned}$$

Conversely, assume that there exists a surjective mapping $\alpha : B \to A$ such that α is an isometry function and $(\min B)\alpha = \min A$. We divide our proof into two cases. Firstly, we assume that B is a finite set. Since α is surjective and B is finite, we can write

$A = \{a_1, a_2, \dots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$

and $B = \{b_1, b_2, \dots, b_n\}$ where $b_1 < b_2 < \cdots < b_n$

for some $n \in \mathbb{N}$. By assumption, we have $b_1\alpha = (\min B)\alpha = \min A = a_1$. Let $x \in B$. Since α is an isometry function, we get that

$$\begin{aligned}x - b_1 &= |x - b_1| \\ &= |x\alpha - b_1\alpha| \\ &= x\alpha - a_1.\end{aligned}$$

Thus $x\alpha = a_1 - b_1 + x$. Set $C = \{-a_1 + b_1\} \in \mathbb{S}$. Let $b \in B$. Then $b\alpha = a_1 - b_1 + b$. Hence $b = (-a_1 + b_1) + b\alpha \in C * A$. Thus $B \subseteq C * A$. Let $y \in C * A$. Then $y = \min C + a$ for some $a \in A$. From α is surjective, there exists $z \in B$ such that $z\alpha = a$. Since $z\alpha = a_1 - b_1 + z$, we obtain that

$$\begin{aligned}y &= \min C + a \\ &= (-a_1 + b_1) + z\alpha \\ &= (-a_1 + b_1) + a_1 - b_1 + z \\ &= z \in B.\end{aligned}$$

We see that $C * A \subseteq B$. Hence $B = C * A \in \mathbb{S} * A$.

Now, we consider B as an infinite set. Then $(\min B)\alpha = \min A$. Similarly way as above, we can prove that $B \in L_A$.

The following examples show that each condition in Theorem 23 can not be omitted.

Example 24. Let $A = \{1,2,3\}$ and $B = \{1,2,5\}$. We suppose that $B \in L_A$. Then $B = C * A$ for some $C \in \mathbb{S}$. This implies that $\min B = \min(C * A) = \min C + \min A$. Since $\min A = \min B = 1$, we have $\min C = 0$. Therefore

$$\begin{aligned} B &= \{\min C + 1, \min C + 2, \min C + 3\} \\ &= \{1,2,3\} \neq B \end{aligned}$$

which is a contradiction. We list all surjections from B onto A as follow that

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix} \text{ and } \alpha_6 = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 2 & 1 \end{pmatrix}.$$

None of them is isometry. Hence B does not satisfy Theorem 23.

Example 25. Let $A = \{1,2,10\}$ and $B = \{5,13,14\}$. We assume that $B \in L_A$, then $B = C * A$ for some $C \in \mathbb{S}$. This implies that

$$\min B = \min(C * A) = \min C + \min A.$$

Since $\min A = 1$ and $\min B = 5$, we obtain that $\min C = 4$. This implies that

$$\begin{aligned} B = C * A &= \{\min C + 1, \min C + 2, \min C + 10\} \\ &= \{5,6,14\} \neq B \end{aligned}$$

which is a contradiction. Thus $B \notin L_A$. We list all surjections from B onto A as follow that

$$\beta_1 = \begin{pmatrix} 5 & 13 & 14 \\ 1 & 2 & 10 \end{pmatrix}, \beta_2 = \begin{pmatrix} 5 & 13 & 14 \\ 1 & 10 & 2 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 5 & 13 & 14 \\ 2 & 10 & 1 \end{pmatrix}, \beta_4 = \begin{pmatrix} 5 & 13 & 14 \\ 10 & 2 & 1 \end{pmatrix},$$

$$\beta_5 = \begin{pmatrix} 5 & 13 & 14 \\ 2 & 1 & 10 \end{pmatrix} \text{ and } \beta_6 = \begin{pmatrix} 5 & 13 & 14 \\ 10 & 1 & 2 \end{pmatrix}.$$

There is only one isometry function which is a bijection. That is, β_4. But $(\min B)\beta_4 \neq \min A$. This implies that B dose not satisfy Theorem 23.

It is easy to see that the inverse of an isometry function is also isometry.

Proposition 26. Let α be a bijection from B onto A where $A, B \in \mathbb{S}$. If α is an isometry function and $(\min B)\alpha = \min A$, then there exists a bijection $\beta : A \to B$ such that β is an isometry function and $(\min A)\beta = \min B$.

Theorem 27. Let $A, B \in \mathbb{S}$. Then $L_A \cap L_B \neq \emptyset$ if and only if $L_A = L_B$.

Proof. Assume that $L_A \cap L_B \neq \emptyset$. Then there exists $C \in L_A \cap L_B$ such that $C \in L_A$ and $C \in L_B$. From Theorem 23 and Proposition 26, there exists a bijection $\alpha : A \to C$ such that α is isometry and $(\min A)\alpha = \min C$. Similarly, there exists a bijective isometry $\beta : C \to B$ such that $(\min C)\beta = \min B$. Therefore $\alpha\beta$ is a bijection from A onto B. We will show that $L_A \subseteq L_B$. Let $X \in L_A$. By Theorem 23, there exists $\alpha_1 : X \to A$ such that α_1 is an isometry bijection and $(\min X)\alpha_1 = \min A$. So $\alpha_1\alpha\beta : X \to B$ is also bijective. To verify that $\alpha_1\alpha\beta$ is an isometry function, let $x, y \in X$. This implies that

$$\begin{aligned} |x\alpha_1\alpha\beta - y\alpha_1\alpha\beta| &= |x\alpha_1\alpha - y\alpha_1\alpha| \\ &= |x\alpha_1 - y\alpha_1| \\ &= |x - y|. \end{aligned}$$

Moreover, $(\min X)\alpha_1\alpha\beta = \min A$. Hence $X \in L_A$ via Theorem 23. Similarly, we can show that $L_B \subseteq L_A$ whence $L_A = L_B$.

Theorem 28. Let $A, B \in \mathbb{S}$. Then $A * B = B * A$ if and only if $L_A = L_B$.

Proof. Suppose that $A * B = B * A$. Since $A * B \in L_B$ and $B * A \in L_A$, we have $A * B = B * A \in L_A \cap L_B$. So $L_A \cap L_B \neq \emptyset$. By Theorem 27, we have $L_A = L_B$. Assume that $L_A = L_B$. It follows from Corollary 19 that $A * B = B * A$.

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References

Backelin, J. (1990). On the number of semigroups of natural numbers. *Mathematica Scandinavica*, 66, 197-215.

Barucci, V. (2009). On propinquity of numerical semigroups and one-dimensional local Cohen Macaulay rings. *Proceedings of Fifth International Fez Conference on Commutative Algebra and applications*. 49-60.