A characterization of clean matrices in $M_3(z)$

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Abstract

An $n\times n$ matrix over a commutative ring with identity is clean if it is the sum of an idempotent matrix and a unit. In 2009, Rajeswari and Aziz gave necessary and sufficient criteria for a matrix in $M_2(\mathbb{Z})$ to be clean and discussed the involved Diophantine equations. In this paper, we extend those results to a larger set, $M_3(\mathbb{Z})$. We characterize when a matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\in M_3(\mathbb{Z})$$

is clean. As its application, we discuss the relation between clean matrices and the existence of non-trivial solution of certain types of Diophantine equations.

Keywords: clean matrix, idempotent matrix, Diophantine equation

Introduction

Let R be a commutative ring with identity and $M_n(R)$ be a set of all $n\times n$ matrices over R. Recall that a matrix $A\in M_n(R)$ is clean if it is the sum of an idempotent matrix $E\in M_n(R)$ (i.e. $E^2=E$) and a unit $U\in M_n(R)$.

In (Khurana & Lam, 2004), a characterization of 2×2 clean matrices of the form $\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\in M_2(\mathbb{Z})$ has been discussed. Later, in (Rajeswari & Aziz, 2009), necessary and sufficient criteria for a 2×2 matrix $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{Z})$ are derived. These criteria involve the existence of solutions of some types of Diophantine equations.

In this paper, we extend the set of matrices to be $M_3(\mathbb{Z})$. Our main purpose is to determine the cleanness of a 3×3 matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ over $\mathbb{Z}$. First of all, we list all 3×3 idempotent matrices over $\mathbb{Z}$. Second of all, we investigate when a given 3×3 matrix minus an idempotent matrix is a unit. These criteria relate to the existence of solutions of the following Diophantine equations of degree 2 in three variables:

$$M_{31}x^2-M_{13}z^2-M_{32}xy+\left(M_{33}+M_{11}\right)xz+M_{12}yz-M_{31}x+M_{32}y+\left(\det A-M_{33}+1\right)z=0$$

and

$$M_{31}x^2-M_{13}z^2-M_{32}xy+\left(M_{33}+M_{11}\right)xz+M_{12}yz-M_{31}x+M_{32}y+\left(\det A-M_{33}-1\right)z=0$$

where M_{ij} s are minors of A. It is easy to see that $x=1,y=0,z=0$ and $x=0,y=0,z=0$ are solutions of these equations. We shall call them as *trivial solutions*. A *non-trivial solution* occurs when $z\neq 0, z\mid x(1-x)$ and $z\mid y(1-x)$. Third of all, we show that if one of the above equations admits a non-trivial solution, then the matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ is clean.

If $E\in M_3(\mathbb{Z})$ is idempotent, then $\det E=0$ or $\det E=1$. We shall call a clean matrix $E+U$ that 1-clean if $\det E=1$, and call that 0-clean if $\det E=0$.

Throughout the paper, we consider 3×3 matrices over $\mathbb{Z}$. Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$, E be an idempotent matrix, U be a unit, and M_{ij} s be minors of A.

Main Results

The following lemma is the characterization of all idempotents in $M_3(\mathbb{Z})$.

Lemma 1. $E=\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\in M_3(\mathbb{Z})$ is idempotent if and only if it is one of the following forms:

$$\begin{pmatrix} e_{11} & 0 & e_{13} \\ \frac{e_{23}e_{11}}{e_{13}} & 0 & e_{23} \\ \frac{e_{11}(1-e_{11})}{e_{13}} & 0 & 1-e_{11} \end{pmatrix},\begin{pmatrix} e_{11} & 0 & e_{13} \\ -\frac{e_{23}(1-e_{11})}{e_{13}} & 1 & e_{23} \\ \frac{e_{11}(1-e_{11})}{e_{13}} & 0 & 1-e_{11} \end{pmatrix},\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ 0 & 0 & 0 \\ \frac{e_{11}(1-e_{11})}{e_{13}} & \frac{e_{12}(1-e_{11})}{e_{13}} & 1-e_{11} \end{pmatrix},\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ 0 & 1 & 0 \\ \frac{e_{11}(1-e_{11})}{e_{13}} & -\frac{e_{11}e_{12}}{e_{13}} & 1-e_{11} \end{pmatrix},$$

$$\begin{pmatrix} e_{11} & e_{12} & 0 \\ \frac{e_{11}(1-e_{11})}{e_{12}} & 1-e_{11} & 0 \\ \frac{e_{11}e_{32}}{e_{12}} & e_{32} & 0 \end{pmatrix},\begin{pmatrix} e_{11} & e_{12} & 0 \\ \frac{e_{11}(1-e_{11})}{e_{12}} & 1-e_{11} & 0 \\ -\frac{e_{32}(1-e_{11})}{e_{12}} & e_{32} & 1 \end{pmatrix},\begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1-e_{33} & e_{23} \\ \frac{e_{21}e_{33}}{e_{23}} & \frac{e_{33}(1-e_{33})}{e_{23}} & e_{33} \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1-e_{33} & e_{23} \\ -\frac{e_{21}(1-e_{33})}{e_{23}} & \frac{e_{33}(1-e_{33})}{e_{23}} & e_{33} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ e_{31} & 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{21}e_{32} & e_{32} & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{31} & e_{32} & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ -e_{21}e_{32} & e_{32} & 1 \end{pmatrix},\begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Let $E=\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\in M_3(\mathbb{Z})$. If E is idempotent, then $E^2=E$ i.e.

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}=\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}.$$

Then we get the following nine equations:

$$e_{11}e_{11}+e_{12}e_{21}+e_{13}e_{31}=e_{11} \quad \text{\_\_\_\_\_\_\_\_\_\_(1)}$$
$$e_{11}e_{12}+e_{12}e_{22}+e_{13}e_{32}=e_{12} \quad \text{\_\_\_\_\_\_\_\_\_\_(2)}$$
$$e_{11}e_{13}+e_{12}e_{23}+e_{13}e_{33}=e_{13} \quad \text{\_\_\_\_\_\_\_\_\_\_(3)}$$
$$e_{21}e_{11}+e_{22}e_{21}+e_{23}e_{31}=e_{21} \quad \text{\_\_\_\_\_\_\_\_\_\_(4)}$$
$$e_{21}e_{12}+e_{22}e_{22}+e_{23}e_{32}=e_{22} \quad \text{\_\_\_\_\_\_\_\_\_\_(5)}$$
$$e_{21}e_{13}+e_{22}e_{23}+e_{23}e_{33}=e_{23} \quad \text{\_\_\_\_\_\_\_\_\_\_(6)}$$
$$e_{31}e_{11}+e_{32}e_{21}+e_{33}e_{31}=e_{31} \quad \text{\_\_\_\_\_\_\_\_\_\_(7)}$$
$$e_{31}e_{12}+e_{32}e_{22}+e_{33}e_{32}=e_{32} \quad \text{\_\_\_\_\_\_\_\_\_\_(8)}$$
$$e_{31}e_{13}+e_{32}e_{23}+e_{33}e_{33}=e_{33}\,. \quad \text{\_\_\_\_\_\_\_\_\_\_(9)}$$

Suppose that $e_{13}\neq 0$ and $e_{12}=0$. Putting $e_{12}=0$ in (3), we get $1-e_{11}=e_{33}$. Putting $e_{12}=0$ in (2), we get $e_{32}=0$. Putting $e_{12}=0$ in (1), we get $e_{31}=\dfrac{e_{11}(1-e_{11})}{e_{13}}$. Putting $e_{12}=0$ and $e_{32}=0$ in (5), we get $e_{22}=0$ or $e_{22}=1$. If $e_{22}=0$ then from (4) we obtain that $E=\begin{pmatrix} e_{11} & 0 & e_{13} \\ \frac{e_{23}e_{11}}{e_{13}} & 0 & e_{23} \\ \frac{e_{11}(1-e_{11})}{e_{13}} & 0 & 1-e_{11} \end{pmatrix}$. If $e_{22}=1$ then from (6) we obtain that $E=\begin{pmatrix} e_{11} & 0 & e_{13} \\ -\frac{e_{23}(1-e_{11})}{e_{13}} & 1 & e_{23} \\ \frac{e_{11}(1-e_{11})}{e_{13}} & 0 & 1-e_{11} \end{pmatrix}$.

Similarly, suppose that $e_{13}\neq 0$ and $e_{23}=0$. We obtain that

$$E=\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ 0 & 0 & 0 \\ \frac{e_{11}(1-e_{11})}{e_{13}} & \frac{e_{12}(1-e_{11})}{e_{13}} & 1-e_{11} \end{pmatrix} \text{ or } E=\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ 0 & 1 & 0 \\ \frac{e_{11}(1-e_{11})}{e_{13}} & -\frac{e_{11}e_{12}}{e_{13}} & 1-e_{11} \end{pmatrix}.$$

Similarly, suppose that $e_{13}=0$ and $e_{12}\neq 0$. We obtain that

$$E=\begin{pmatrix} e_{11} & e_{12} & 0 \\ \frac{e_{11}(1-e_{11})}{e_{12}} & 1-e_{11} & 0 \\ \frac{e_{11}e_{32}}{e_{12}} & e_{32} & 0 \end{pmatrix} \text{ or } E=\begin{pmatrix} e_{11} & e_{12} & 0 \\ \frac{e_{11}(1-e_{11})}{e_{12}} & 1-e_{11} & 0 \\ -\frac{e_{32}(1-e_{11})}{e_{12}} & e_{32} & 1 \end{pmatrix}.$$

Similarly, suppose that $e_{13}=0$ and $e_{23}\neq 0$. We obtain that

$$E=\begin{pmatrix} 0 & 0 & 0 \\ e_{21} & 1-e_{33} & e_{23} \\ \frac{e_{21}e_{33}}{e_{23}} & \frac{e_{33}(1-e_{33})}{e_{23}} & e_{33} \end{pmatrix} \text{ or } E=\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1-e_{33} & e_{23} \\ -\frac{e_{21}(1-e_{33})}{e_{23}} & \frac{e_{33}(1-e_{33})}{e_{23}} & e_{33} \end{pmatrix}.$$

Now, suppose that $e_{13}=0, e_{12}=0$ and $e_{23}=0$. Replacing them in (1) yields ${e_{11}}^2=e_{11}$, that is $e_{11}=0$ or $e_{11}=1$. Replacing them in (5) yields ${e_{22}}^2=e_{22}$, that is $e_{22}=0$ or $e_{22}=1$. Replacing them in (9) yields ${e_{33}}^2=e_{33}$, that is $e_{33}=0$ or $e_{33}=1$. Therefore,

$$E=\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\e_{21}&0&0\\e_{31}&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\e_{21}&1&0\\e_{21}e_{32}&e_{32}&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\e_{31}&e_{32}&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\e_{31}&e_{32}&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\e_{21}&0&0\\-e_{21}e_{32}&e_{32}&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\e_{21}&1&0\\e_{31}&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}.$$

Finally, suppose that $e_{13}\neq 0, e_{12}\neq 0$ and $e_{23}\neq 0$. The matrix is not idempotent.

Conversely, it is easy to check that if E is any one of those matrices in the statement then E is idempotent. This completes the proof. ■

Now, we shall discuss a necessary and sufficient condition of a matrix $A\in M_3(\mathbb{Z})$ to be 1-clean. From Lemma 1, we can conclude that $\det E=1$ if and only if $E=I$. Therefore, A is 1-clean if and only if $A-I$ is a unit.

Theorem 2. A is 1-clean if and only if $\det A+\operatorname{tr}A-M_{11}-M_{12}-M_{13}=0$ or 2.

Proof. Since $\det U\cdot\det U^{-1}=\det\left(UU^{-1}\right)=\det I=1$ and $\det U, \det U^{-1}$ must be integers, then $\det U=\pm 1$. Therefore,

A is 1-clean $\Leftrightarrow A-I=U$

$\Leftrightarrow \det(A-I)=\det U$

$$\Leftrightarrow \begin{vmatrix} a_{11}-1 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{31} & a_{32} & a_{33}-1 \end{vmatrix}=\pm 1$$

$\Leftrightarrow a_{11}a_{22}a_{33}-a_{11}a_{33}-a_{33}a_{22}+a_{33}-a_{12}a_{22}+a_{11}+a_{22}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}$
$-a_{31}a_{13}a_{22}+a_{13}a_{31}-a_{32}a_{23}a_{11}+a_{23}a_{32}-a_{21}a_{12}a_{33}+a_{21}a_{12}-1=1$ or -1

$\Leftrightarrow a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{13}a_{22}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}+a_{11}$
$+a_{22}+a_{33}-\left(a_{22}a_{33}-a_{32}a_{23}\right)-\left(a_{11}a_{33}-a_{31}a_{13}\right)-\left(a_{12}a_{22}-a_{12}a_{21}\right)=0$ or 2

$\Leftrightarrow \det A+\operatorname{tr}A-M_{11}-M_{12}-M_{13}=0$ or 2. ■

Example 3. Consider the matrix $A=\begin{pmatrix} 2&-2&8\\0&0&0\\0&0&2 \end{pmatrix}$. We can see that $A=E+U$, where $E=\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ and $U=\begin{pmatrix} 1&-2&8\\0&-1&0\\0&0&1 \end{pmatrix}$.

Since $\det E=1$, A is 1-clean.

An easier way is to check that

$$\det A+\operatorname{tr}A-M_{11}-M_{22}-M_{33}=0+4-0-4=0.$$

By Theorem 3, A is 1-clean. ■

Next, we will consider the case that $\det E=0$. From Lemma 1, there are 15 cases that $\det E=0$, but we will consider here just some of them because the results for the other cases are similar.

Theorem 4. (i) If $E=\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ A is a unit.

(ii) If $E=\begin{pmatrix} 1&0&x\\0&0&0\\0&0&0 \end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ $\det A-M_{11}-M_{13}x=\pm 1$.

(iii) If $E=\begin{pmatrix}0&0&x\\0&0&0\\0&0&1\end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ $\det A-M_{33}-M_{13}x=\pm1$.

(iv) If $E=\begin{pmatrix}1&x&0\\0&0&0\\0&0&0\end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ $\det A-M_{11}+M_{12}x=\pm1$.

(v) If $E=\begin{pmatrix}0&x&0\\0&1&0\\0&0&0\end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ $\det A-M_{22}+M_{12}x=\pm1$.

(vi) If $E=\begin{pmatrix}0&0&0\\0&0&x\\0&0&1\end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ $\det A-M_{33}+M_{23}x=\pm1$.

(vii) If $E=\begin{pmatrix}0&0&0\\0&1&x\\0&0&0\end{pmatrix}$ then $A-E$ is a unit $\Leftrightarrow$ $\det A-M_{22}+M_{23}x=\pm1$.

Proof. (i) Obvious.

(ii) Let $E=\begin{pmatrix}1&0&x\\0&0&0\\0&0&0\end{pmatrix}$. Then

$A-E$ is a unit $\Leftrightarrow A-E=U$

$\Leftrightarrow \det(A-E)=\det U$

$$\Leftrightarrow \begin{vmatrix}a_{11}-1&a_{12}&a_{13}-x\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=\pm1$$

$$\Leftrightarrow a_{11}a_{22}a_{33}-a_{22}a_{33}+a_{11}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{12}a_{32}x-a_{31}a_{22}a_{13}+a_{31}a_{22}x$$
$$-a_{32}a_{23}a_{11}+a_{32}a_{23}-a_{33}a_{21}a_{12}=\pm1$$

$$\Leftrightarrow a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{13}a_{22}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}+a_{32}a_{23}$$
$$-a_{22}a_{33}+(a_{31}a_{22}-a_{21}a_{32})x=\pm1$$

$\Leftrightarrow \det A-M_{11}-M_{13}x=\pm1$.

The proofs of (iii), (iv), (v), (vi), and (vii) are similar to the proof of (ii). ∎

Theorem 5. Let $E=\begin{pmatrix}x&y&z\\0&0&0\\\frac{x(1-x)}{z}&\frac{y(1-x)}{z}&1-x\end{pmatrix}\in M_3(\mathbb{Z})$ with $z\neq0, z\mid x(1-x)$ and $z\mid y(1-x)$. Then $\det E=0$. Moreover, $A-E$ is a unit if and only if one of the following Diophantine equations

(i) $M_{31}x^2-M_{13}z^2-M_{32}xy+(M_{33}+M_{11})xz+M_{12}yz-M_{31}x+M_{32}y+(\det A-M_{33}+1)z=0$

(ii) $M_{31}x^2-M_{13}z^2-M_{32}xy+(M_{33}+M_{11})xz+M_{12}yz-M_{31}x+M_{32}y+(\det A-M_{33}-1)z=0$

has (x,y,z) as a non-trivial solution.

Proof. Let $E=\begin{pmatrix}x&y&z\\0&0&0\\\frac{x(1-x)}{z}&\frac{y(1-x)}{z}&1-x\end{pmatrix}$. It is clear that $\det E=0$. Moreover,

$A-E$ is a unit $\Leftrightarrow A-E=U$

$\Leftrightarrow \det(A-E)=\det U$

$$\Leftrightarrow \begin{vmatrix}a_{11}-x&a_{12}-y&a_{13}-z\\a_{21}&a_{22}&a_{23}\\a_{31}-\frac{x(1-x)}{z}&a_{32}-\frac{y(1-x)}{z}&a_{33}-(1-x)\end{vmatrix}=\pm1$$

$$\Leftrightarrow a_{11}a_{22}x - a_{22}a_{33}x + a_{23}a_{32}x - a_{12}a_{21}x - a_{23}a_{31}y + a_{21}a_{33}y - a_{21}a_{32}z + a_{22}a_{31}z - a_{23}a_{12}\frac{x}{z}$$
$$+ a_{23}a_{12}\frac{x^2}{z} - a_{21}a_{13}\frac{y}{z} + a_{21}a_{13}\frac{xy}{z} + a_{23}a_{11}\frac{y}{z} - a_{23}a_{11}\frac{xy}{z} + a_{22}a_{13}\frac{x}{z} - a_{22}a_{13}\frac{x^2}{z} + a_{12}a_{21}$$
$$- a_{11}a_{22} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{13}a_{22} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \pm 1 = 0$$
$$\Leftrightarrow (a_{11}a_{22} - a_{12}a_{21})xz + (a_{23}a_{32} - a_{22}a_{33})xz + (a_{21}a_{33} - a_{23}a_{31})yz + (a_{22}a_{31} - a_{21}a_{32})z^2$$
$$+ (a_{22}a_{13} - a_{23}a_{12})(x - x^2) + (a_{23}a_{11} - a_{21}a_{13})(y - xy)$$
$$+ (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{13}a_{22} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12})z$$
$$+ (a_{12}a_{21} - a_{11}a_{22})z \pm z = 0$$
$$\Leftrightarrow M_{31}x^2 - M_{13}z^2 - M_{32}xy + (M_{33} + M_{11})xz + M_{12}yz - M_{31}x + M_{32}y + (\det A - M_{33} \pm 1)z = 0.$$

■

Theorem 6. A matrix $A \in M_3(\mathbb{Z})$ is 0-clean if and only if one of the following conditions is satisfied:

(i) A is a unit.

(ii) $\det A - M_{11} - M_{13}x = \pm 1$ for some x.

(iii) $\det A - M_{33} - M_{13}x = \pm 1$ for some x.

(iv) $\det A - M_{11} + M_{12}x = \pm 1$ for some x.

(v) $\det A - M_{22} + M_{12}x = \pm 1$ for some x.

(vi) $\det A - M_{33} + M_{23}x = \pm 1$ for some x.

(vii) $\det A - M_{22} + M_{23}x = \pm 1$ for some x.

(viii) The Diophantine equation

$$M_{31}x^2 - M_{13}z^2 - M_{32}xy + (M_{33} + M_{11})xz + M_{12}yz - M_{31}x + M_{32}y + (\det A - M_{33} + 1)z = 0$$

has a non-trivial solution.

(ix) The Diophantine equation

$$M_{31}x^2 - M_{13}z^2 - M_{32}xy + (M_{33} + M_{11})xz + M_{12}yz - M_{31}x + M_{32}y + (\det A - M_{33} - 1)z = 0$$

has a non-trivial solution.

Proof. Conditions (i) to (vii) in Theorem 4 are precisely conditions (i) to (vii) in Theorem 6. In addition, conditions (viii) and (ix) in Theorem 6 are obtained from Theorem 5. ■

Example 7. Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. We can see that $A = E + U$, where $E = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

Since $\det E = 0$, A is 0-clean.

An easier way is to put $x = -2$ and check that

$$\det A - M_{11} - M_{13}x = 2 - 1 - 0(-2) = 1.$$

By Theorem 6 (ii), A is 0-clean. ■

Corollary 8. Let $E = \begin{pmatrix} 0 & y & z \\ 0 & 0 & 0 \\ 0 & \frac{y}{z} & 1 \end{pmatrix} \in M_3(\mathbb{Z})$ with $z \neq 0$ and $z \mid y$. Then $\det E = 0$. Moreover, $A - E$ is a unit if and only if one of the following Diophantine equations

(i) $-M_{13}z^2 + M_{12}yz + M_{32}y + (\det A - M_{33} + 1)z = 0$

(ii) $-M_{13}z^2 + M_{12}yz + M_{32}y + (\det A - M_{33} - 1)z = 0$

has (x, y, z) as a non-trivial solution.

Proof. It follows from Theorem 5 by replacing $x = 0$. ■

Corollary 9. Let $E=\begin{pmatrix} x & 0 & z \\ 0 & 0 & 0 \\ \frac{x(1-x)}{z} & 0 & 1-x \end{pmatrix}\in M_3(\mathbb{Z})$ with $z\neq 0$ and $z\mid x(1-x)$. Then $\det E=0$. Moreover, $A-E$ is a unit if and only if one of the following Diophantine equations

(i) $M_{31}x^2-M_{13}z^2+(M_{33}+M_{11})xz-M_{31}x+(\det A-M_{33}+1)z=0$

(ii) $M_{31}x^2-M_{13}z^2+(M_{33}+M_{11})xz-M_{31}x+(\det A-M_{33}-1)z=0$

has (x,y,z) as a non-trivial solution.

Proof. It follows from Theorem 5 by replacing $y=0$. ■

Corollary 10. Let $E=\begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\in M_3(\mathbb{Z})$ with $z\neq 0, z\mid x(1-x)$ and $z\mid y(1-x)$. Then $\det E=0$. Moreover, $A-E$ is a unit if and only if one of the following Diophantine equations

(i) $-M_{13}z^2+(\det A-M_{33}+1)z=0$

(ii) $-M_{13}z^2+(\det A-M_{33}-1)z=0$

has (x,y,z) as a non-trivial solution.

Proof. It follows from Theorem 5 by replacing $x=y=0$. ■

Conclusion

In this work, we aim to determine the cleanness of a matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\in M_3(\mathbb{Z})$. Firstly, Lemma 1 gives us all idempotent matrices E in $M_3(\mathbb{Z})$. Secondly, the condition to be 1-clean is given in Theorem 2. After that, we study the conditions that $A-E$ is a unit for several matrices E in Theorem 4, and also study the relation between clean matrices and the existence of solutions of some Diophantine equations in Theorem 5. Finally, we derive Theorem 6 as our main result. It shows us the condition to be 0-clean, which involve that $A-E$ is a unit and the existence of non-trivial solutions of the following Diophantine equations of degree 2 in three variables:

$$M_{31}x^2-M_{13}z^2-M_{32}xy+(M_{33}+M_{11})xz+M_{12}yz-M_{31}x+M_{32}y+(\det A-M_{33}\pm 1)z=0$$

where M_{ij} s are minors of A.

The Diophantine equations derived here are more complicated than and cannot be reduced to those derived in [3].

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